

COMPLEXITY: Exercise No. 2
partial solutions

1 Question 1

For each of the following statements, prove, disprove or show that it is an open problem:

1. If $L_1, L_2 \in \mathbf{coNP}$ then $L_1 \cap L_2 \in \mathbf{coNP}$. (Test 99)

Answer: True. $L_1^c, L_2^c \in \mathbf{NP}$, so $L_1^c \cup L_2^c \in \mathbf{NP}$, and hence $L_1 \cap L_2 = (L_1^c \cup L_2^c)^c \in \mathbf{coNP}$.

2. If $L \in \mathbf{NP}$, $L_1 \subsetneq L$ and $L_1 \in \mathbf{coNP}$ then $L - L_1 \in \mathbf{NP}$.

Answer: True. $L_1 \in \mathbf{coNP}$ so $L_1^c \in \mathbf{NP}$, and since \mathbf{NP} is closed for intersection, $L - L_1 = L \cap L_1^c \in \mathbf{NP}$.

3. If $L \in \mathbf{NPC}$ then $\{xx : x \in L\} \in \mathbf{NPC}$. (Test 94)

Answer: True. $\{xx : x \in L\}$ is clearly in \mathbf{NP} - check that the input is of the form xx and run the NDTM of L on x . On the other hand, there is a reduction from L , by just duplicating the input, so it is in \mathbf{NPC} .

2 Question 3

Consider the following problem:

SUBGRAPH ISOMORPHISM (SGI):

Instance: Two graphs, $G = (V, E)$, $H = (U, F)$.

Question: Is there a 1-1 mapping $\phi : U \rightarrow V$, such that if $(u, u') \in F$ then $(\phi(u), \phi(u')) \in E$?

Prove that $\mathbf{UHC} \prec \mathbf{SGI}$.

Answer: This is almost reduction by generalization. Observe that \mathbf{UHC} is a special case of \mathbf{SGI} , with H being just a simple cycle of length $|V|$. For this case, the questions asked by \mathbf{SGI} and \mathbf{UHC} are equivalent.

The reduction is, therefore, for an instance $G = (V, E)$ of \mathbf{UHC} , we produce the instance (G, H) , where H is a simple cycle of length $|V|$.

This is clearly a polynomial time reduction.

The reduction is correct, because, by the definitions:

(G, H) is a “yes” instance of $\mathbf{SGI} \Leftrightarrow$

G has a subgraph which is a simple cycle of length $|V| \Leftrightarrow$

G has a Hamiltonian cycle \Leftrightarrow

G is a “yes” instance of \mathbf{UHC} .

3 Question 4

Consider the following problems:

3-SAT:

Instance: A CNF Boolean formula Φ in which each clause has 3 literals.

Question: Is there an assignment that satisfies Φ ?

BALANCED 3-SAT:

Instance: A CNF Boolean formula Φ in which each clause has 3 literals, and every variable in Φ appears negated and unnegated the same number of times.

Question: Is there an assignment that satisfies Φ ?

Prove that 3-SAT \prec BALANCED 3-SAT.

Answer: We count the number of positive and negative occurrences of every variable x_i . If it is not equal, we add clauses of the form $(x_i \vee z \vee \bar{z})$ or $(\bar{x}_i \vee z \vee \bar{z})$ where z is a new variable.

4 Question 5

Consider the following problems:

SAT:

Instance: A CNF Boolean formula Φ .

Question: Is there an assignment that satisfies Φ ?

1/2 SAT: Instance: A CNF formula Φ .

Question: Is there an assignment that satisfies Φ in which exactly half of the variables are TRUE?

Prove that SAT \prec 1/2 SAT.

Answer: Just add n clauses, each containing a new variable and its negation.