COMPLEXITY: Exercise No. 4
partial solutions

1 Question 2

(Test 93) Is the following problem **NP**-complete?
IS-CLIQUE:

Instance: An undirected graph G and a positive integer number k.
Question: Does G contain a clique of size k *or* an independent set of size k?
Answer: The problem is Clearly in **NP**. We show it is **NP**-complete by the following reduction from IS. Given a graph $G = (V, E)$ and a number k, we construct a graph G' by adding $n = |V|$ isolated vertices and setting $k' = n + k$.
Obviously G' does not contain a clique of size greater than n. Thus, it contains a clique or an independent set of size k' $\Leftrightarrow$ it contains an independent set of size k' $\Leftrightarrow$ G contains an independent set of size k.

2 Question 3

(Test 92) Is the following problem **NP**-complete?
DOMINATING SET (DS):

Instance: An undirected graph $G = (V, E)$ and an integer k.
Question: Does G have a dominating set of size $\leq k$? (a dominating set is a set $U \subseteq V$, such that for every $v \in V \setminus U$ there is $u \in U$ such that $(u, v) \in E$)
Answer: We use a reduction from VERTEX COVER. Given an instance $(G = (V, E), k)$ of VC we transform it to an instance $(G' = (V', E'), k)$ of DS as follows. First we remove all isolated vertices from G (clearly, they are not necessary in any VC since they have no edges). G' is similar to G, with a vertex added for each edge, and connected to both sides of the original edge. Formally, V' consists of all the non-isolated vertices in V and a vertex v_e $\forall e \in E$, and E' consists of E, and the new edges (v_e, u) and (v_e, w) $\forall e = (u, w) \in E$. The reduction can clearly be done in polynomial time.

- $\Rightarrow$ Let U be a vertex cover of G. We show that it is also a dominating set in G'. For $\forall v \in V' \setminus U$ there are two possibilities

  1. If $v \in V$ then, since it is not an isolated vertex, there is an edge $(u, v) \in E$. Since U is a vertex cover and $v \notin U$, we have $u \in U$, so v has a neighbour in U.
  2. $v = v_e$ for some edge $e = (u, w) \in E$. Since U is a vertex cover, $u \in U$ or $w \in U$, so v has a neighbour in U.

- $\Leftarrow$ Suppose that U is a dominating set in G' of size $\leq k$. We first get rid of vertices in U corresponding to edges of G. If $v_e \in U$ for some edge $e = (u, w) \in E$, then we replace it with u. Note that:

1. Note that the size of U does not increase in this process.
2. The neighbours of a vertex v_e for $e = (u, w)$ are only u and w. Thus, by removing v_e from U, the only vertices that might become uncovered are v_e, u and w, but they are covered by u. Therefore U remains a dominating set.

$\forall e = (u, w) \in E$, U must contain u or w (otherwise v_e has no neighbour in U). Thus, U is a dominating set in G.

3 Question 4

Is the following problem **NP**-complete?
CONNECTED DOMINATING SET:

Instance: An undirected graph $G = (V, E)$ and a positive integer k.
Question: Does G contain a dominating set S with at most k vertices such that the subgraph of G induced by S (i.e., the graph $G_S = (S, E \cap S \times S)$) is connected?
Answer: We use a reduction from VERTEX COVER (on graphs with at least one edge). Given an instance $(G = (V, E), k)$ of VERTEX COVER we transform it an instance $(G' = (V', E'), k)$ of CONNECTED DOMINATING SET as follows: G' contains the vertices of G, and each two of these vertices are connected by an edge. Furthermore, for each edge (x, y) in E, the graph G' contains a vertex v_{xy} which is connected to x and y. Formally, $V' = V \cup \{v_{xy} | (x, y) \in E\}$, and $E' = \{(x, y) | x, y \in V\} \cup \{(x, v_{xy}), (y, v_{xy}) | (x, y) \in E\}$. The proof of correctness of this reduction is similar the proof of the reduction from VERTEX COVER to DOMINATING SET And since the dominating set we construct consists only of vertices of G, it is connected.

4 Question 5

Is the following problem **NP**-complete?
MINIMUM LEAF SPANNING TREE:

Instance: An undirected graph $G = (V, E)$ and a positive integer k.
Question: Is there a spanning tree for G in which the number of leaves is at most k?
Answer: The problem is obviously in **NP**, since the tree itself is a polynomial witness for membership in this language. The restricted version of this problem, with $k = 2$ is equivalent to the **NP**-complete *Hamiltonian Path* problem, hence, this general problem is also **NP**-complete.

5 Question 6

(Test 95) Is the following problem **NP**-complete?
MAXIMUM LEAF SPANNING TREE:

Instance: An undirected graph G and a positive integer k.
Question: Is there a spanning tree for G in which the number of leaves is at least k?

Answer: We give a reduction from CONNECTED DOMINATING SET: Given an input $(G = (V, E), k)$ to CONNECTED DOMINATING SET, the output of the reduction is $(G, |V| - k)$.

- $\Rightarrow$ Let S be a connected dominating set of G with size at most k. Since $G|_S$ is connected, it has a spanning tree $T_s = (S, E_S)$. We build a spanning tree $T = (V, E_T)$ of G by adding the edges in E_S to E_T. Furthermore, for any vertex $v \in V - S$, we pick a neighbor $u \in S$ and we add the edge (u, v) to E_T. Any vertex $v \in V - S$ is a leaf in T so T has at least $|V| - |S| \geq |V| - k$ leaves.

- $\Leftarrow$ Let $T = (V, E_T)$ be a spanning tree of G with at least $|V| - k$ leaves. Let $S = \{v \in V | v \text{ is not a leaf in } T\}$. S is a dominating set of G (as any leaf in T has a non-leaf neighbor) and $G|_S$ is connected (as $T_{|S}$ is a spanning tree of $G|_S$).