COMPLEXITY: Exercise No. 7
partial solutions

1 Question 1

(Test 98) Is the following problem in **NL**?
Given an undirected graph G, vertices x, y from G, and a positive integer k, does the shortest path from x to y is of length (exactly) k?
Answer: Check, by a variant of the NL algorithm for connectivity, that there is a path of length $\leq k$ between x and y. Then check, by a variant of the NL algorithm for non-connectivity, that there is no path of length $\leq k-1$ between x and y. Answer "Yes" iff both of the above algorithms return "Yes".

2 Question 2

What is the approximation ratio of the greedy algorithm for SET-COVER, when all the sets except one are of size at most k?
Answer: Let $A_1, A_2, ..., A_r$ be the minimal set cover, and let $B_1, B_2, ..., B_t$ be the solution of the greedy algorithm, in the order it added these sets to the set cover. Thus, the only set of size greater then k is B_1. We pay 1 for each set the greedy algorithm adds to the set cover, and hence the total payment is t. We split the payment of B_i between all the elements that are covered after adding B_i, and were not covered before. Denote $C_i = B_i \setminus (B_1 \cup ... \cup B_{i-1})$. Then every $x \in C_i$ pays $c(x) = \frac{1}{|C_i|}$. As shown in class, $\forall 1 \leq i \leq r$, the total payment of the elements in A_i is at most $H(|A_i|)$. If B_1 belongs to the optimal solution, then

$$t-1 = \sum_{x \notin B_1} c(x) \leq \sum_{1 \leq i \leq r : A_i \neq B_1} \sum_{x \in A_i} c(x) \leq \sum_{1 \leq i \leq r : A_i \neq B_1} H(|A_i|) \leq (r-1)H(k).$$

Therefore, $t \leq (r-1)H(k) + 1$. If B_1 does not belong to the optimal solution then

$$t = \sum_x c(x) \leq \sum_{i=1}^{r} \sum_{x \in A_i} c(x) \leq \sum_{i=1}^{r} H(|A_i|) \leq rH(k).$$

In both cases the approximation ratio is at most $H(k) \leq \ln k$.

3 Question 3

Show that the following problem is PSPACE-complete:
Instance: A deterministic TM M and an input x for M.
Question: Does M accepts x without leaving the first $|x|+1$ places of the tape?
Answer: The problem is in PSPACE, because we can simulate M on x using polynomial space. We have to check it does not leave the first $|x|+1$ places of the tape, and count the number of steps to detect an infinite loop. We show that it is PSPACE-complete by a reduction from any problem

in PSPACE. Let $L \in PSPACE$, and let M be a polynomial space TM for L. Suppose M uses $p(n)$ space. Let $\# \notin \Sigma$, and let M' be a TM identical to M, only that it handles #'s as blanks. Given an input x for M, the input to our problem will be M' and $y = x\#^{p(|x|)-|x|}$. M accepts x iff M' accepts y without leaving the first $|y|+1$ places of the tape, and the time of the reduction is $O(p(n))$ (since $|M'|$ is constant and does not depend on the input).

4 Question 4

Find a constant c for which it is NP-hard to approximate VERTEX-COVER to within any constant factor $< c$.

Answer: The reduction from Gap-3-SAT-$[7/8+\epsilon, 1]$ to Clique shows that Gap-Clique-$[7/24+\epsilon, 1/3]$ is NP-hard, and hence the same holds for IS. The reduction from IS to VC thus implies that Gap-VC-$[2/3, 17/24-\epsilon]$ is NP-hard, and therefore it is hard to approximate VC to within any constant factor smaller than $17/16$.