Algorithmic Robotics and Motion Planning

Fall 2006/7

Arrangements, an Overview

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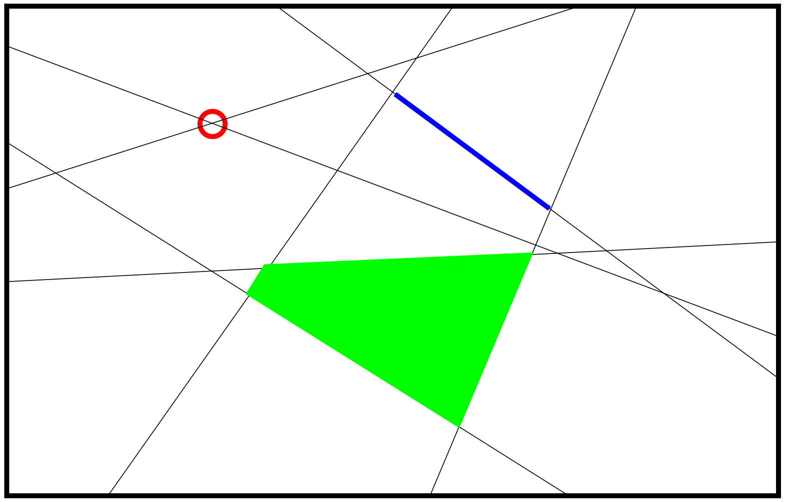
What are arrangements?

Example: an arrangement of lines



vertex

edge

face

What are arrangements, cont'd

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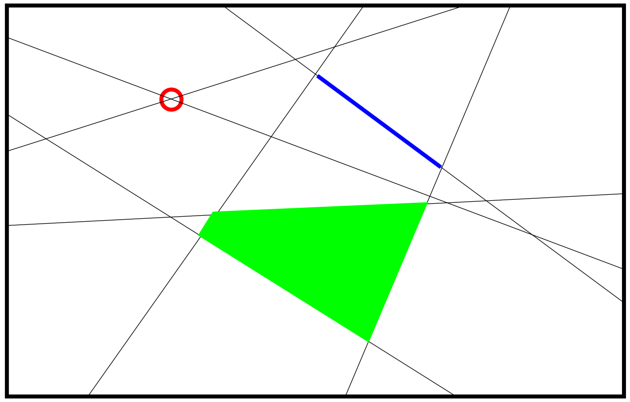
What are arrangements, cont'd

- an arrangement of a set S of geometric objects is the subdivision of space where the objects reside induced by S
- possibly non-linear objects (parabolas), bounded objects (segments, circles), higher dimensions (planes, simplices)
- numerous applications in robotics, molecular biology, vision, graphics, CAD/CAM, statistics, GIS
- have been studied for decades, originally mostly combinatorics
  nowadays mainly studied in combinatorial and computational geometry

Arrangements of lines: Combinatorics



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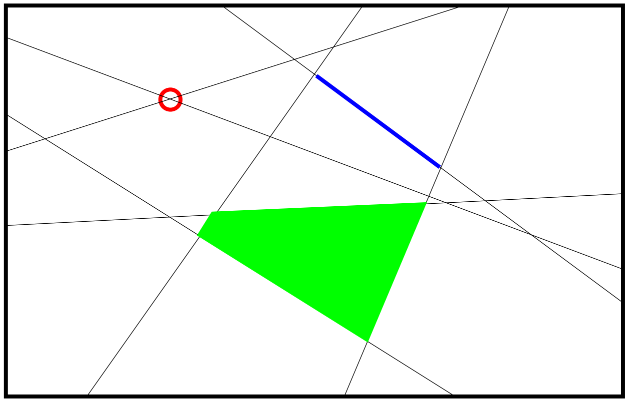

the complexity of an arrangement is the overall number of cells of all dimensions comprising the arrangement

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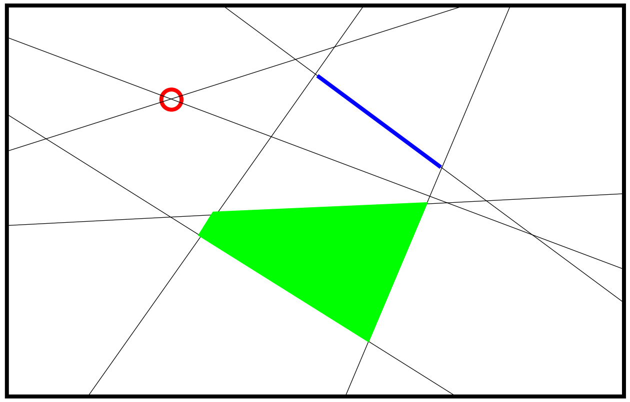

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for planar arrangements we count: vertices, edges, and faces

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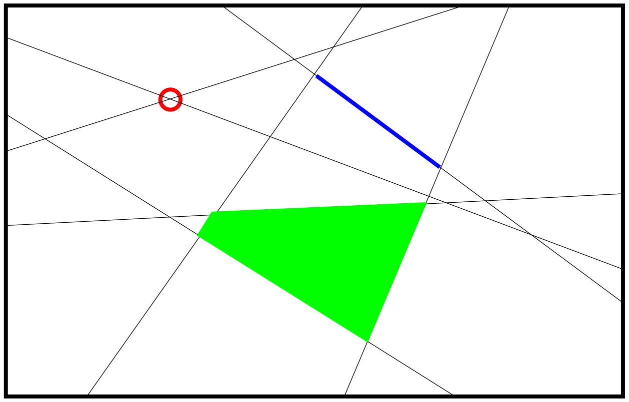

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for planar arrangements we count: vertices, edges, and faces

the general position assumption: two lines meet in a single point, three lines have no point in common

In an arrangements of n lines

number of vertices:

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number of vertices: $n(n-1)/2$

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number of edges:

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In an arrangements of n lines

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number of faces:
using Euler's formula $\#V - \#E + \#F = 2$
we get $n^2/2 + n/2 + 1$

Basic theorem of arrangement complexity

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the maximum combinatorial complexity of an arrangement of n well-behaved (hyper)surfaces in $\mathbb{R}^d$ is $O(n^d)$; there are such arrangements whose complexity is $\Omega(n^d)$

Constructing the arrangement

- representation (data structure)
  - DCEL - the doubly connected edge list

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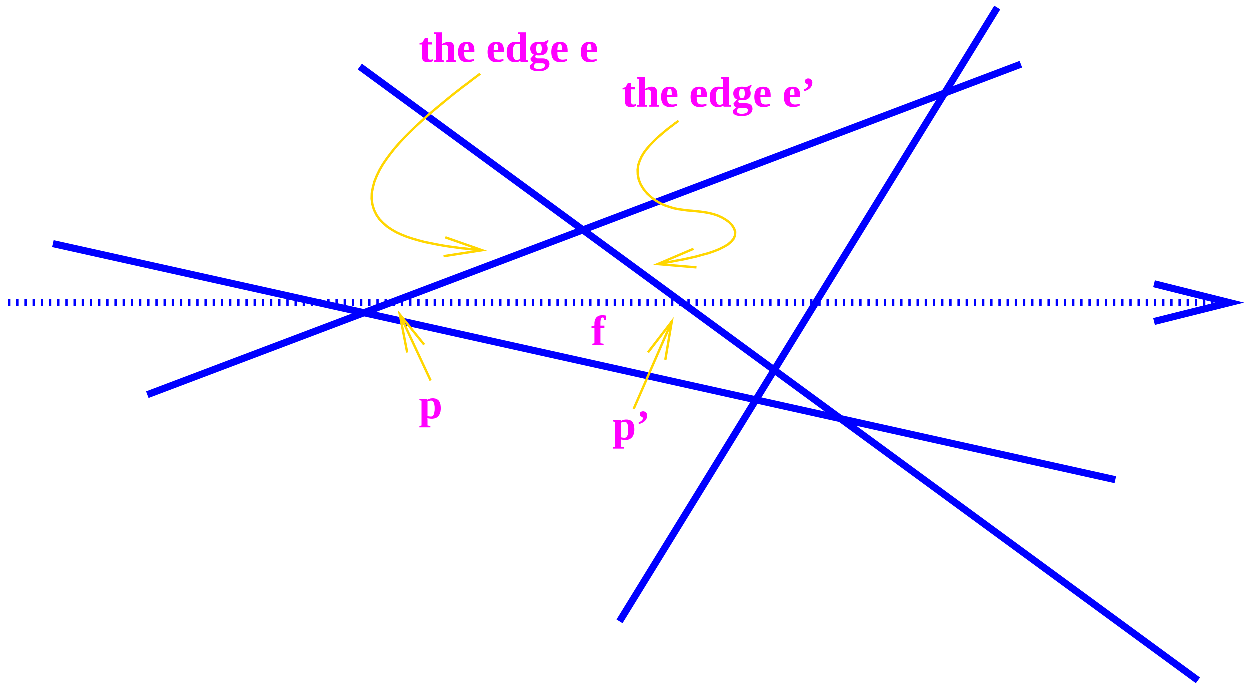
Incremental construction

- computing a bounding box
- inserting the ith line



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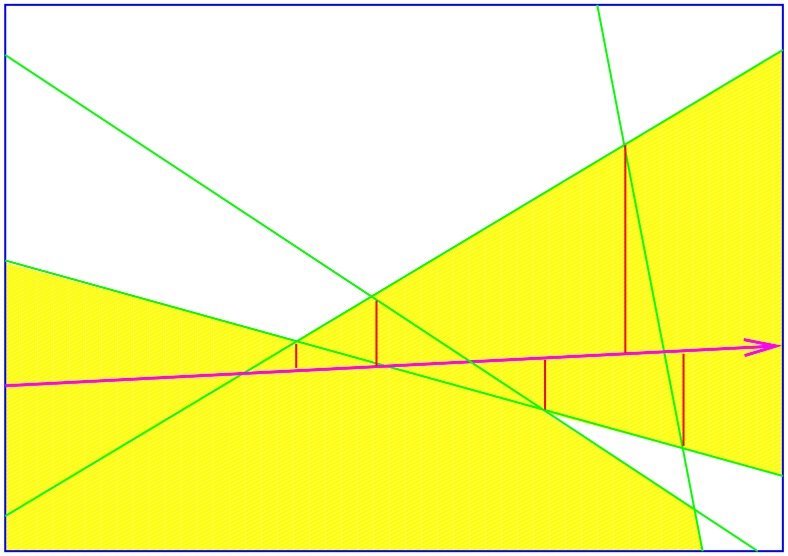
$O($zone complexity$)$

The zone of a curve

the zone of a curve γ in an arrangement $\mathcal{A}$ is the collection of faces of $\mathcal{A}$ intersected by γ



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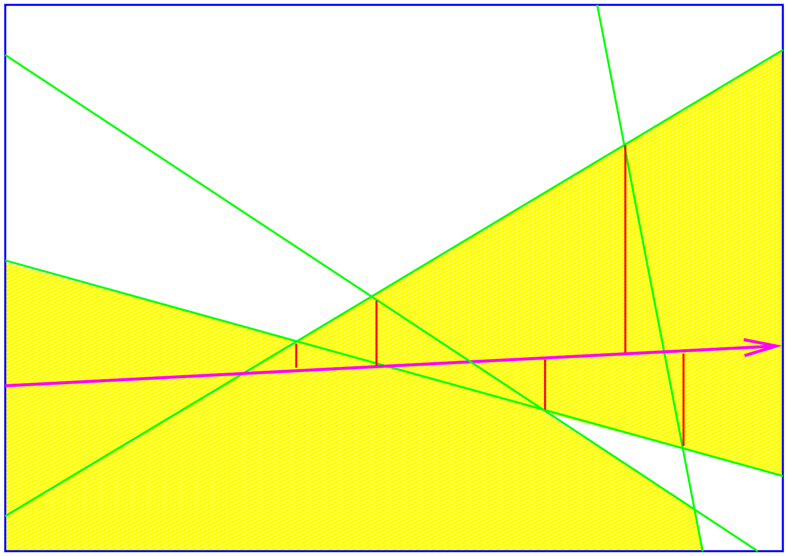

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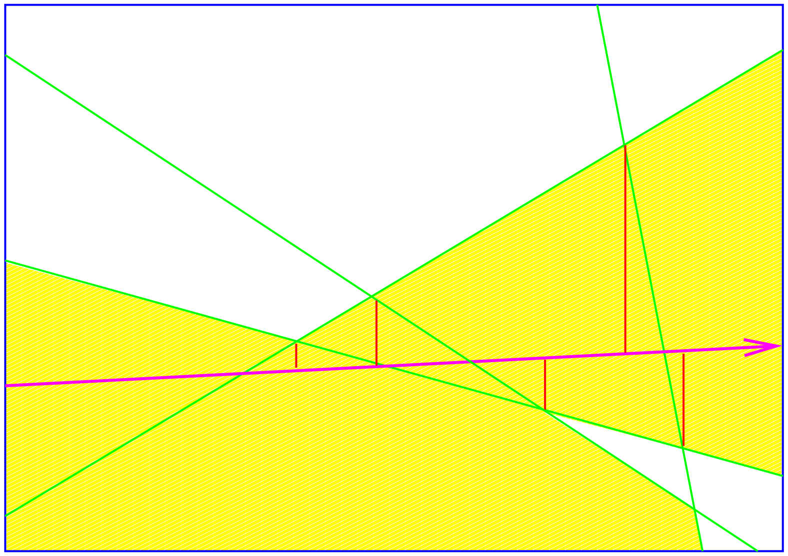

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we need: the complexity of the zone of a line in an arrangement of i lines

Zone theorem

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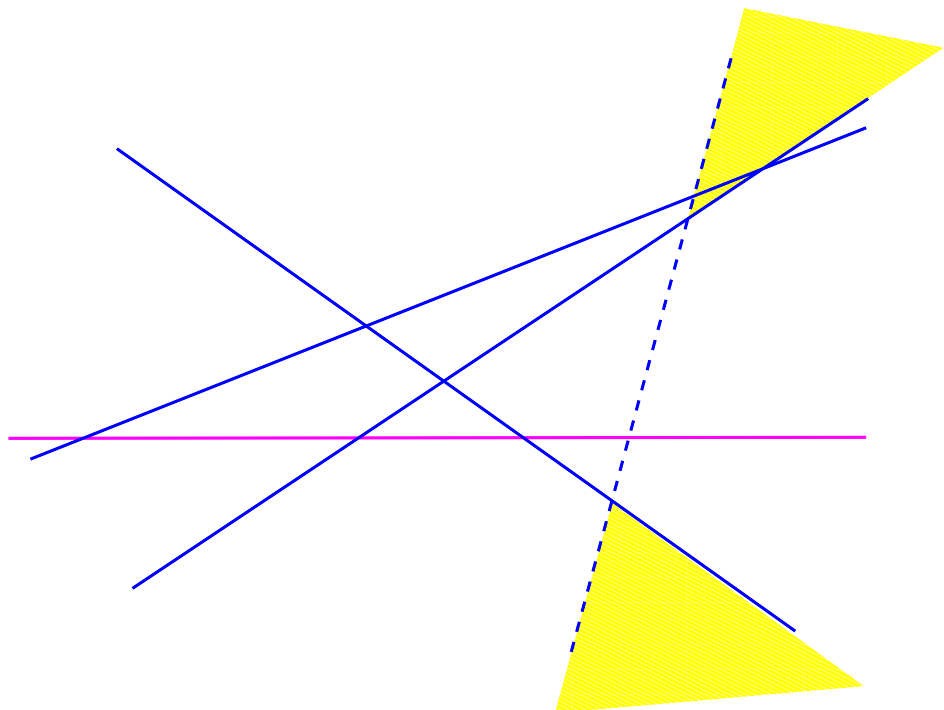

Overall running time of the algorithm

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simple, $O(i)$

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$O($zone complexity$)$ = $O(i)$

overall $O(n^2)$ time

Basics of arrangements, summary

- the combinatorial complexity of an arrangement
- incremental construction
- the zone of an(other) object in an arrangement
- the basic theorem of arrangement complexity
- the real RAM model,the general position assumption

Configuration spaces

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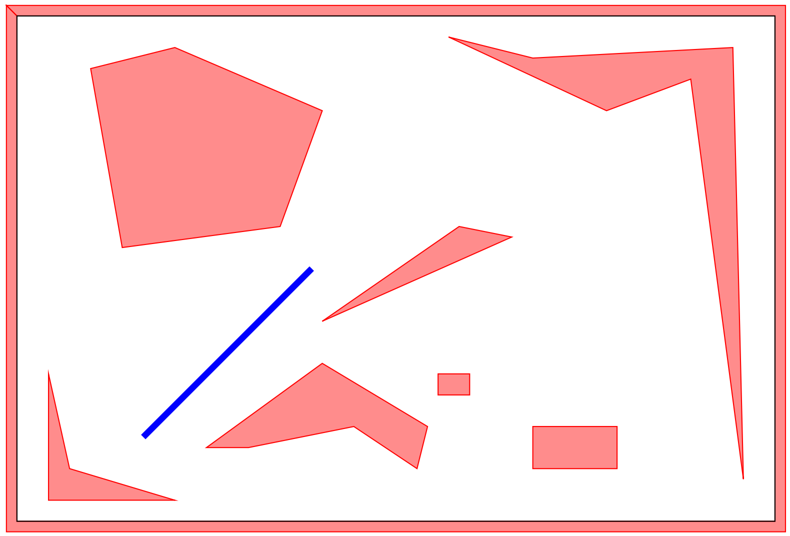
- the property is invariant in each cell of the arrangement

Configuration space for translational motion planning

the rod is translating in the room



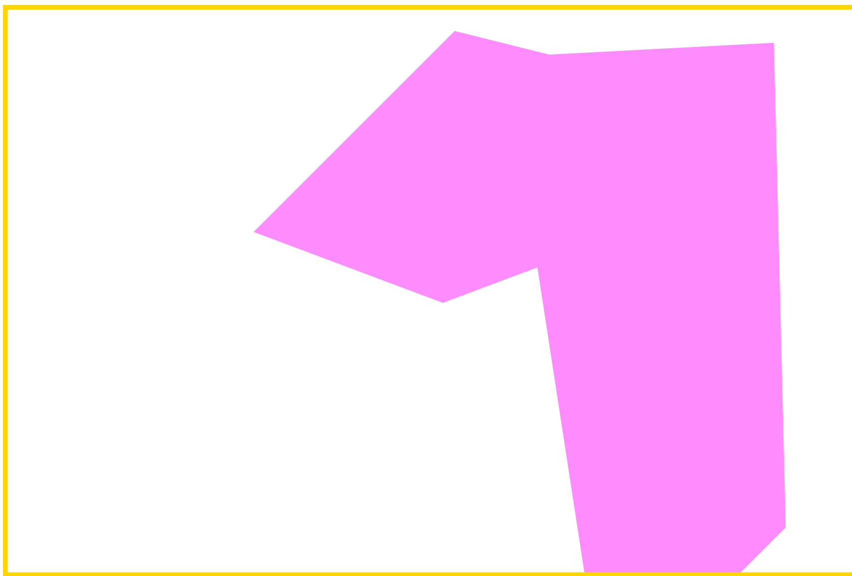
the reference point: the lower end-point of the rod
the configuration space is 2 dimensional

Configuration space obstacles

the robot has shrunk to a point

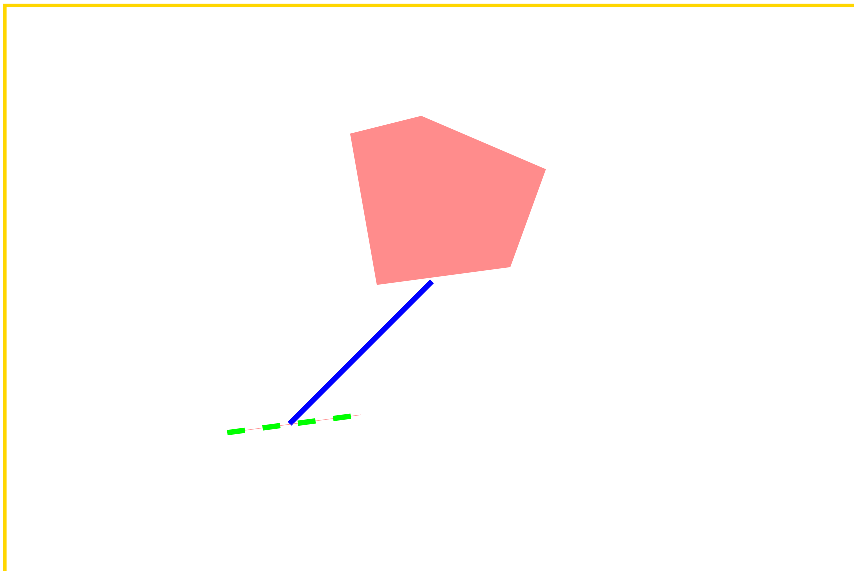
$\Rightarrow$

the obstacles are accordingly expanded

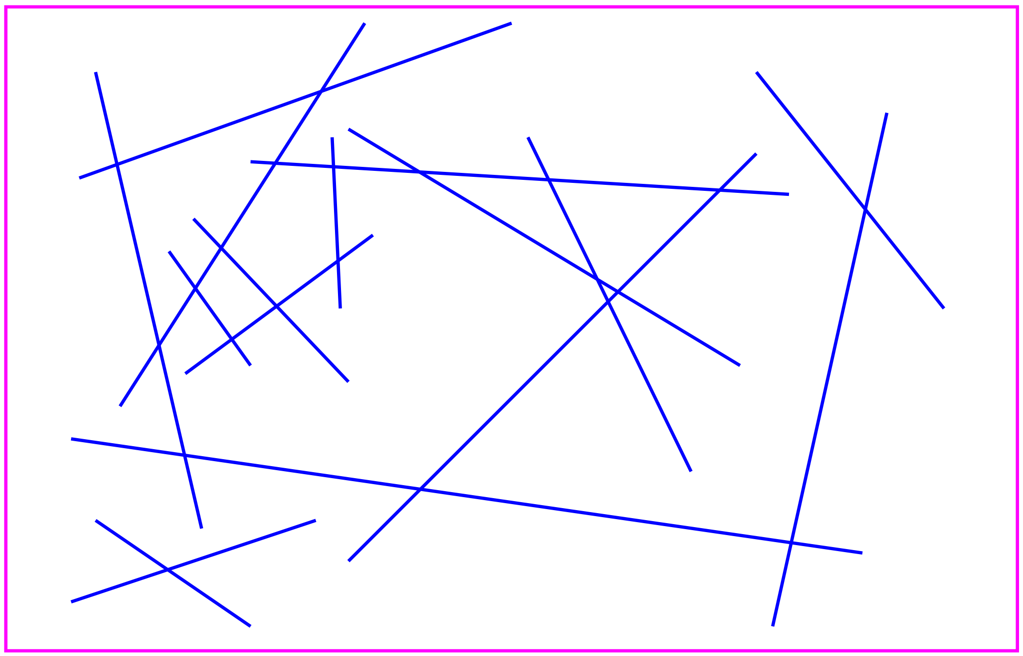


Critical curves in configuration space

the locus of semi-free placements



The arrangement of critical curves in configuration space



Solving a motion-planning problem

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- how complex is the arrangement of the critical curves
- constructing the arrangement and filtering out the forbidden cells
- what is the complexity of the free space
- can we compute the free space efficiently
- do we need to compute the entire free space?

Example: a disc moving among discs

- the critical curves are circles

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Example: an L-shaped robot moving among points

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- how complex is the arrangement of the critical curves?
- what is the complexity of the free space?
- how to compute the free space efficiently?

END