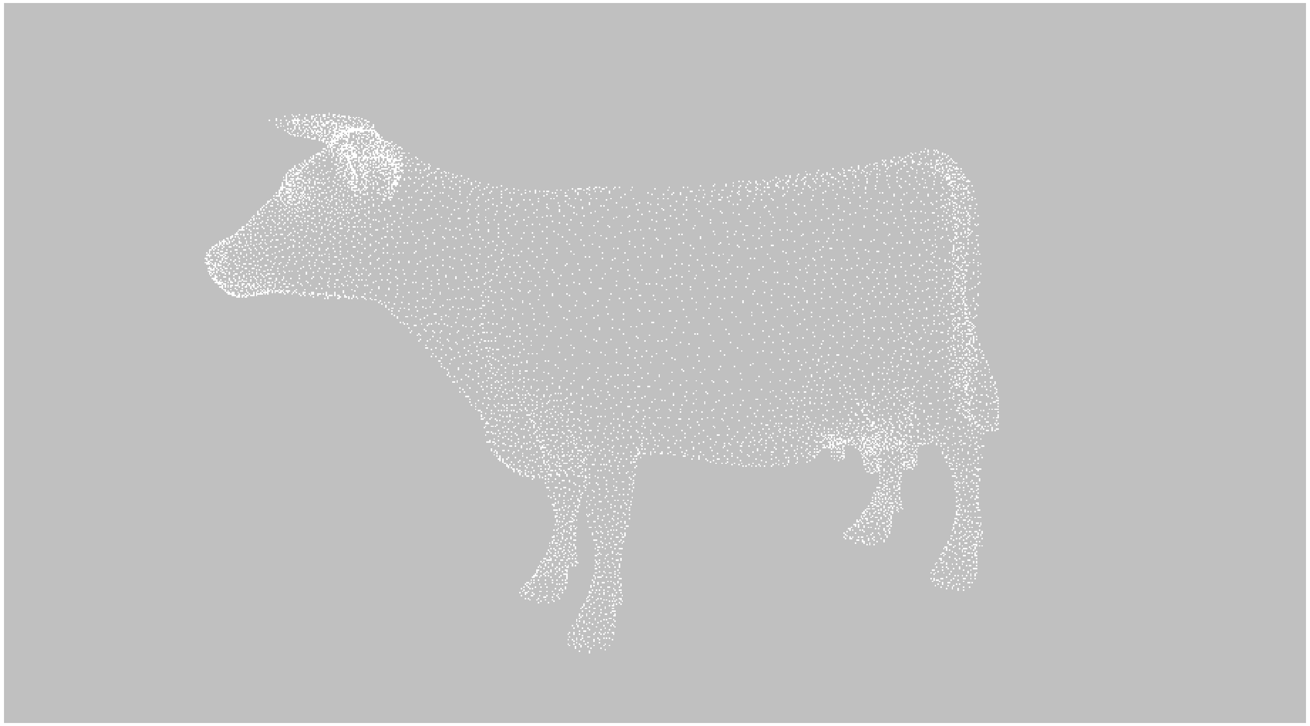
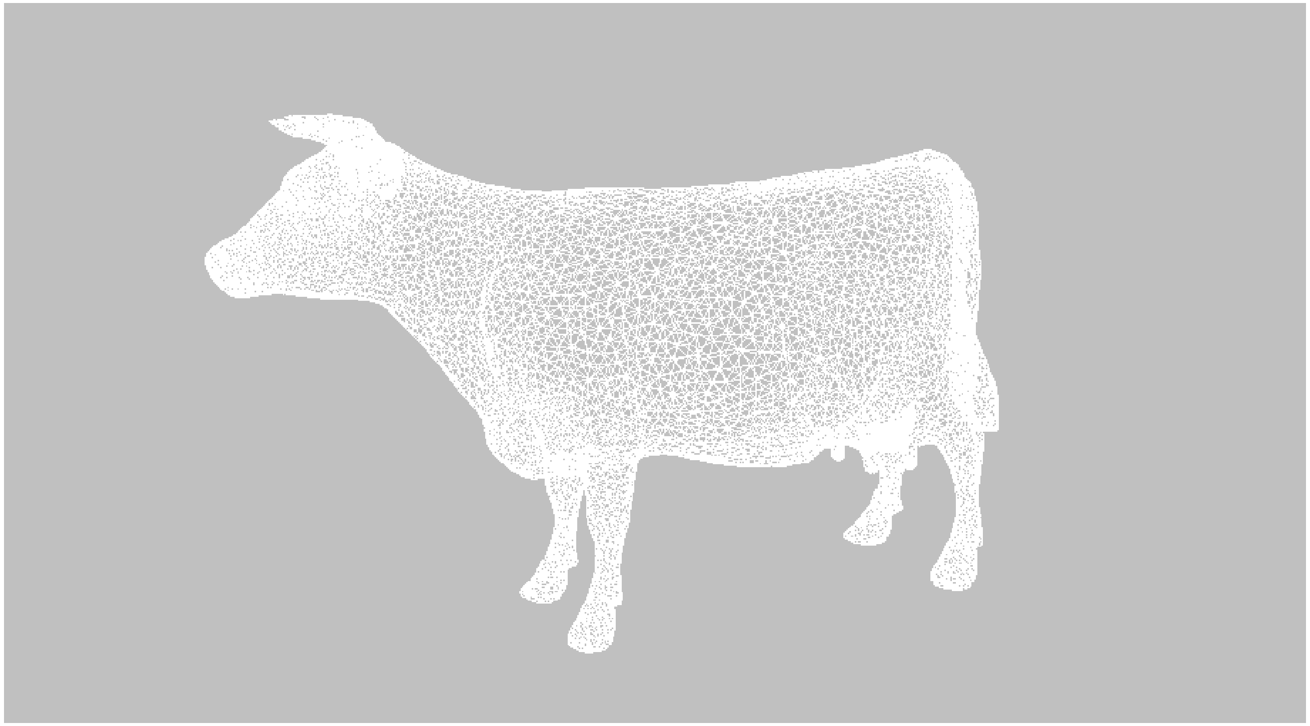
3d Geometry for Computer Graphics

Lesson 1: Basics & PCA

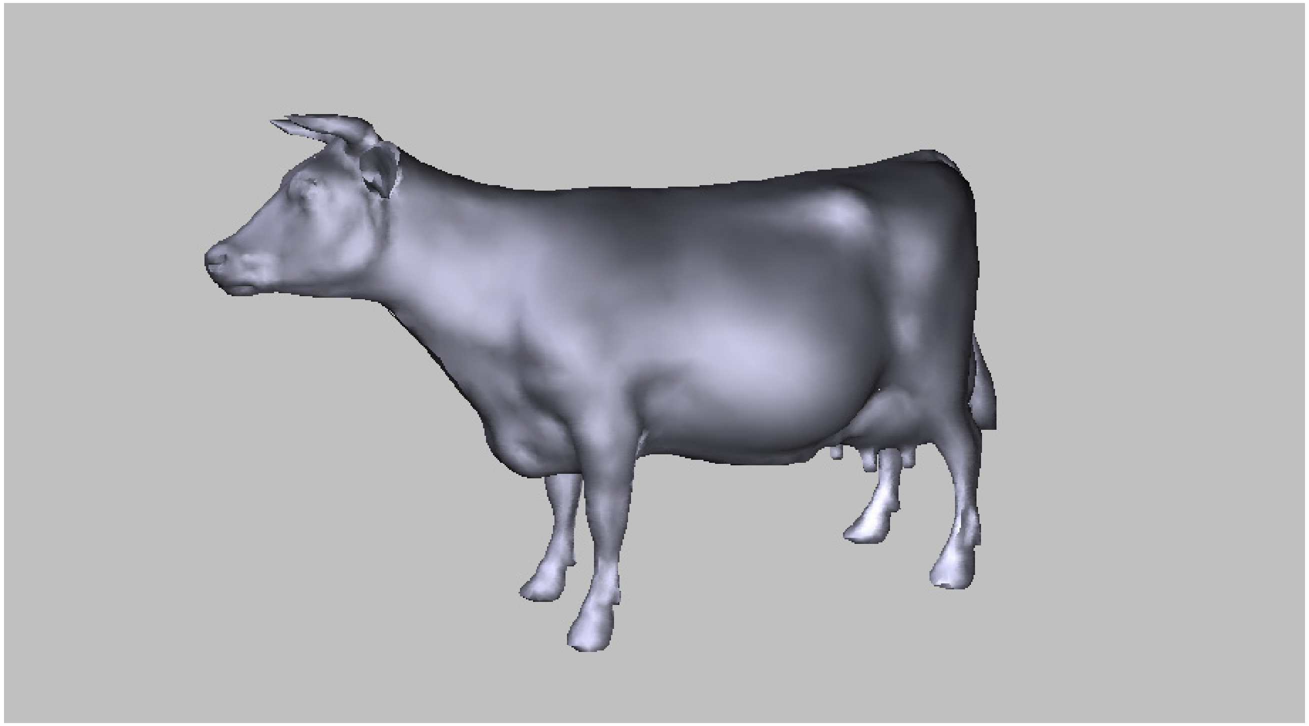
3d geometry



3d geometry

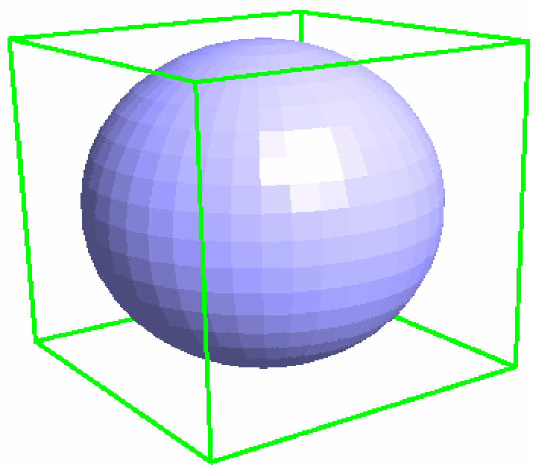


3d geometry



Why??

- We represent objects using mainly linear primitives:
  - points
  - lines, segments
  - planes, polygons
- Need to know how to compute distances, transformations, projections…

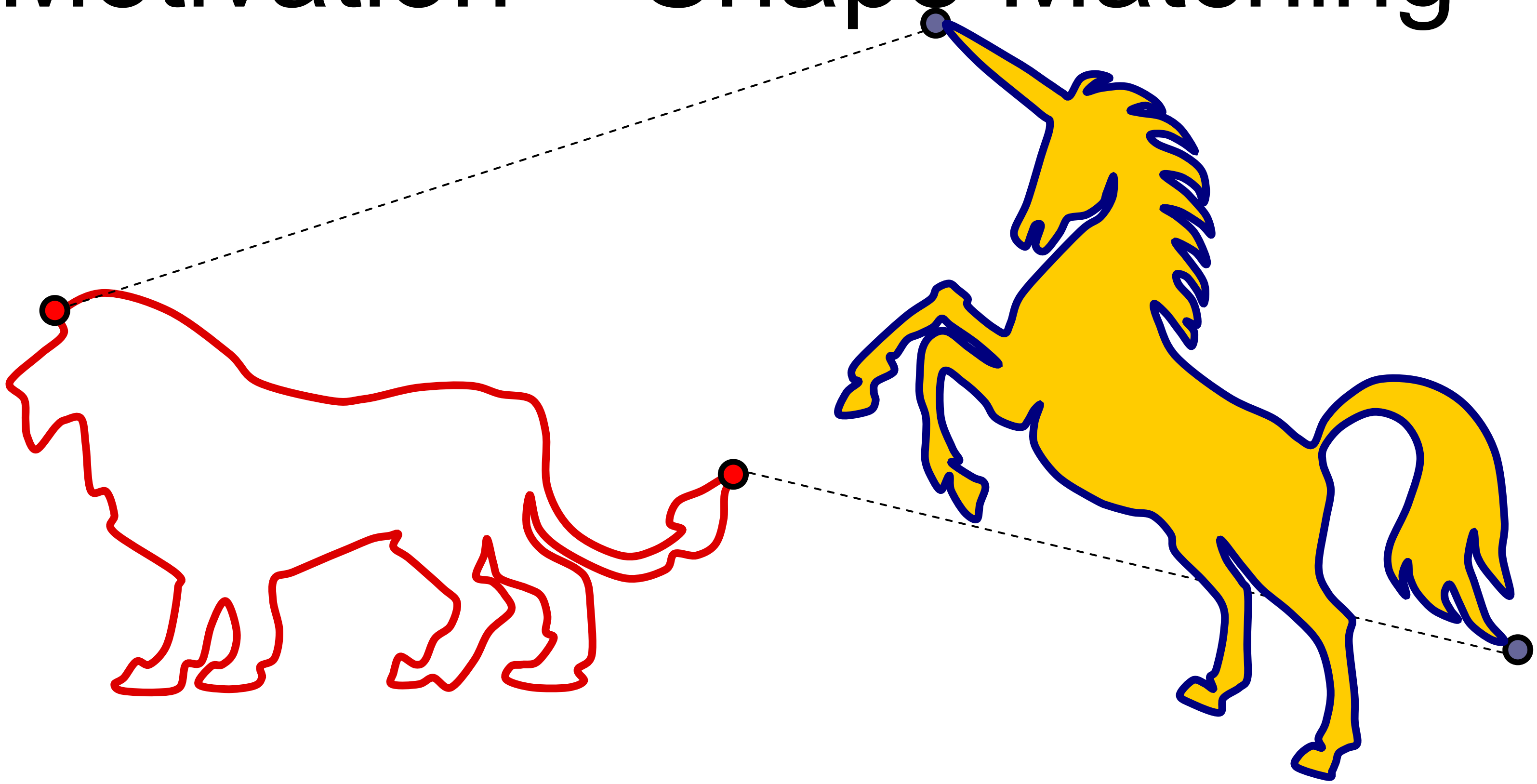


How to approach geometric problems

- We have two ways:
  1. Employ our geometric intuition
  2. Formalize everything and employ our algebra skills

- Often we first do No.1 and then solve with No.2
- For complex problems No.1 is not always easy…

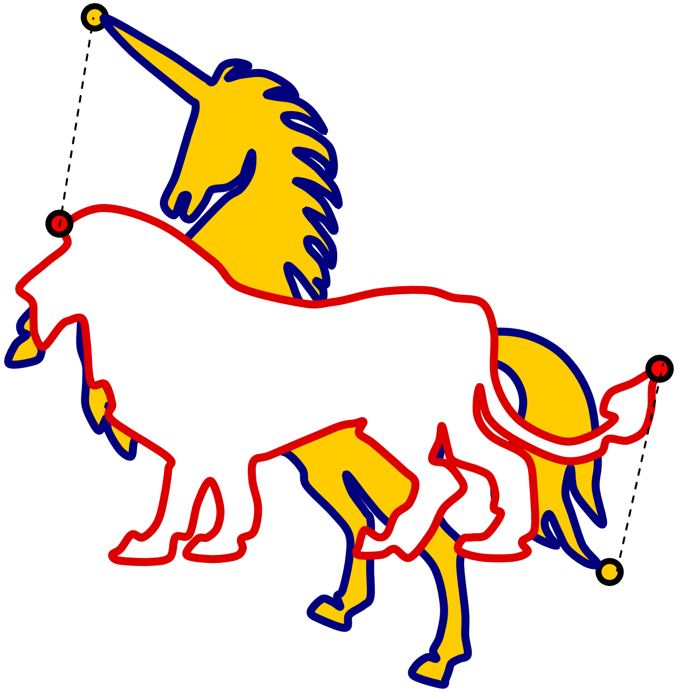
Motivation – Shape Matching



There are tagged feature points in both sets that are matched by the user

What is the best transformation that aligns the unicorn with the lion?

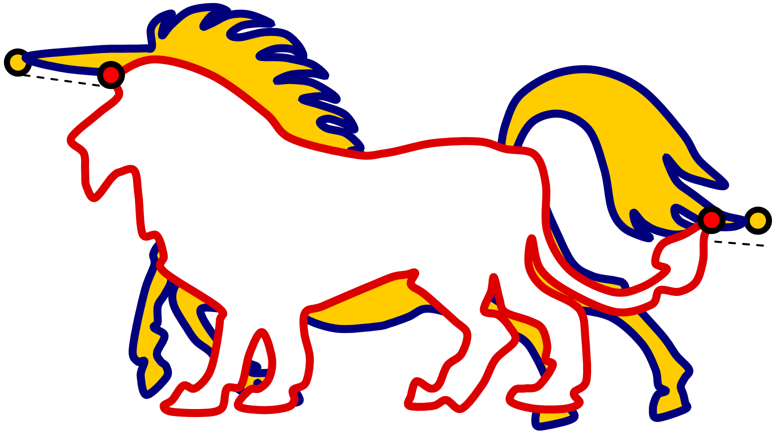
Motivation – Shape Matching



Regard the shapes as sets of points and try to “match” these sets using a *linear transformation*

The above is not a good alignment….

Motivation – Shape Matching

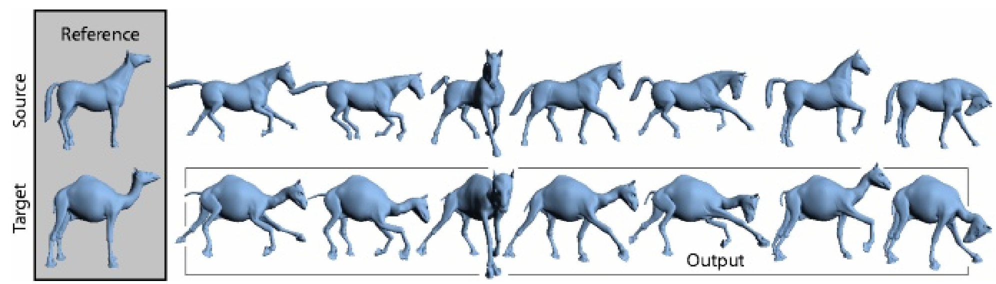


Regard the shapes as sets of points and try to “match” these sets using a *linear transformation*

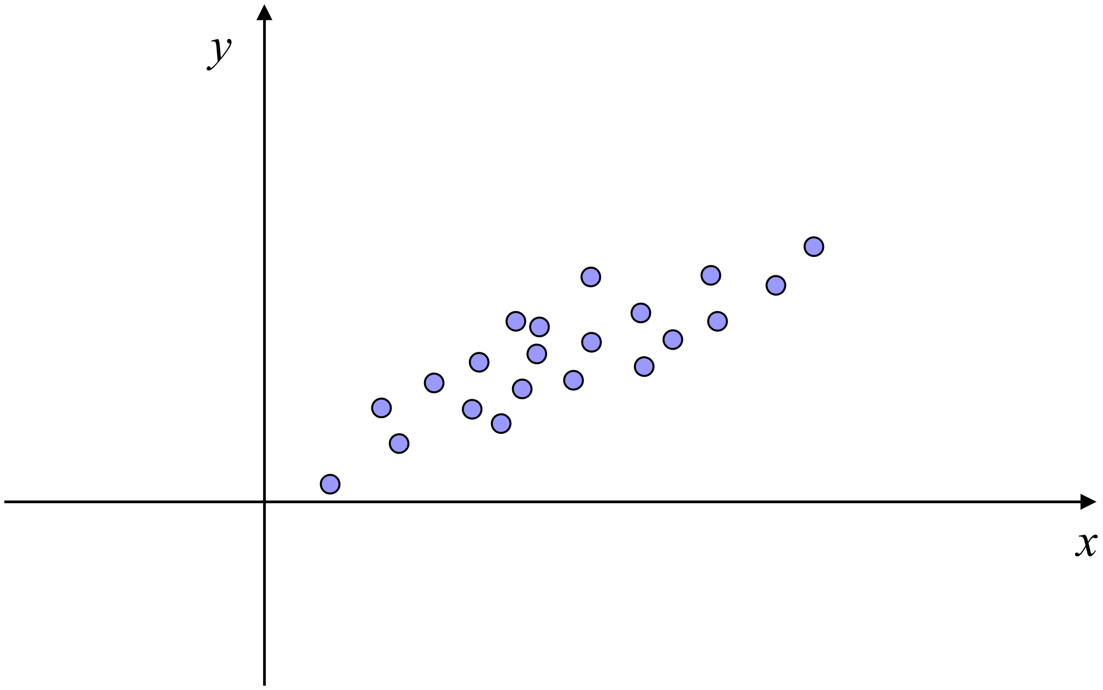
To find the best *rotation* we need to know SVD…

Applications for Shape Matching

- Mesh Comparison
- Deformation Transfer
- Morphing

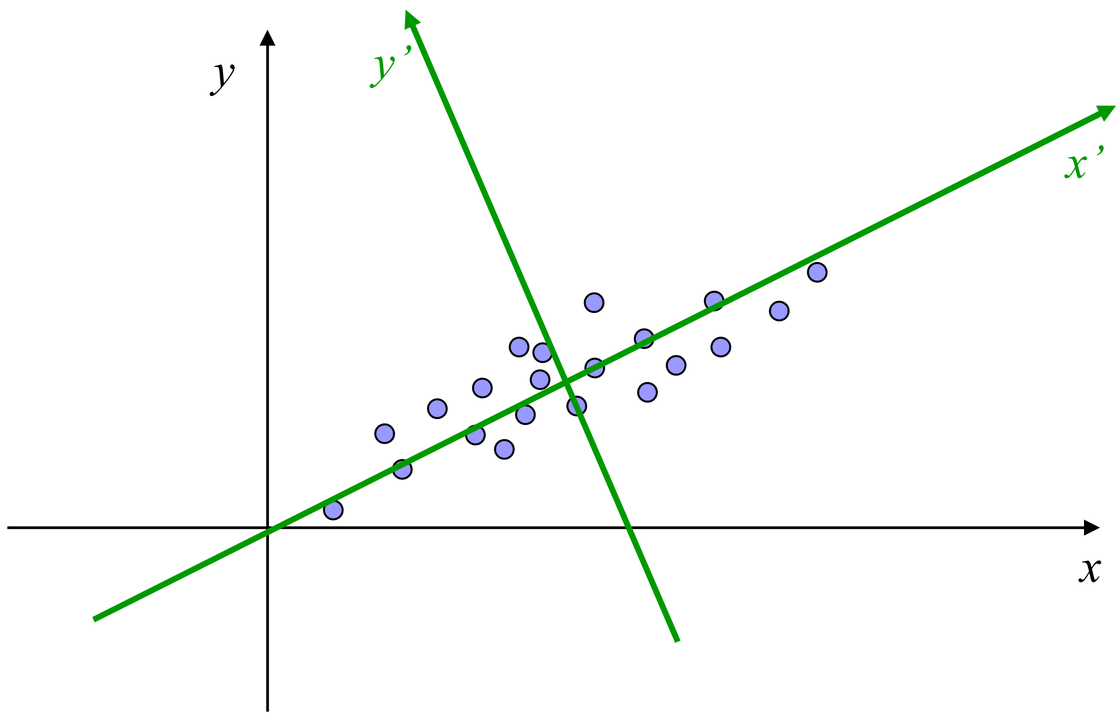


Principal Component Analysis



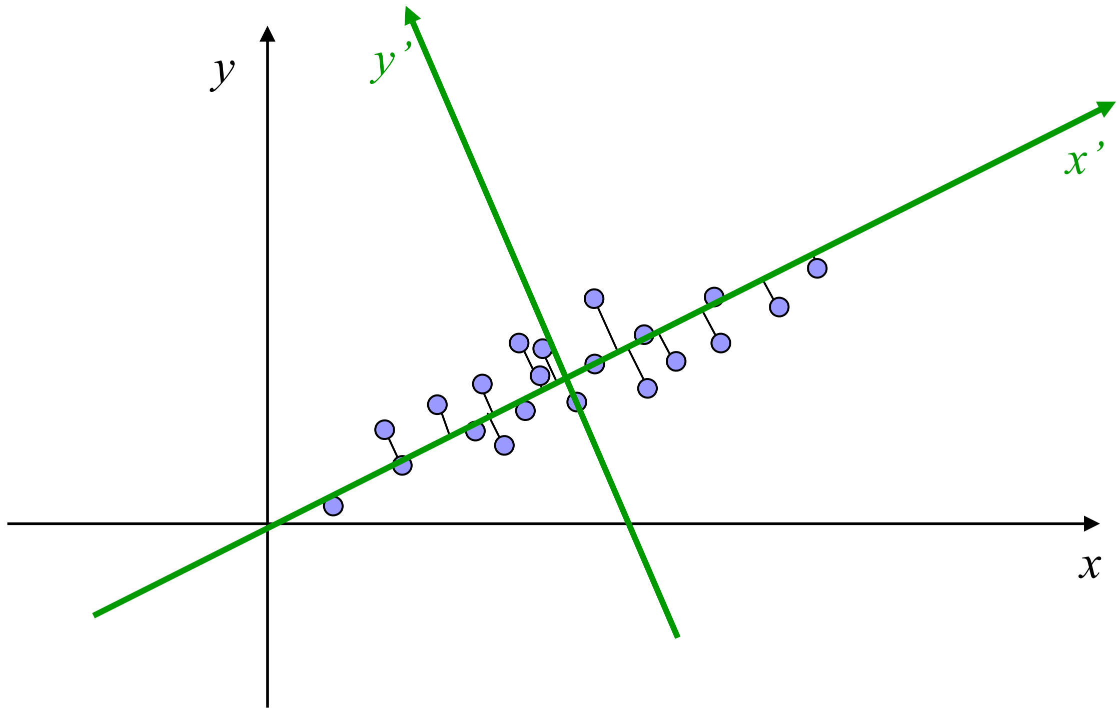
Given a set of points,
find the best line that approximates it

Principal Component Analysis



Given a set of points,
find the best line that approximates it

Principal Component Analysis



When we learn PCA (Principal Component Analysis), we'll know how to find these axes that minimize the sum of distances^2

PCA and SVD

- PCA and SVD are important tools not only in graphics but also in statistics, computer vision and more.

SVD: $A = U\Lambda V^T$

Λ is diagonal contains the singular values of A

- To learn about them, we first need to get familiar with eigenvalue decomposition.
- So, today we'll start with linear algebra basics reminder.

Vector space

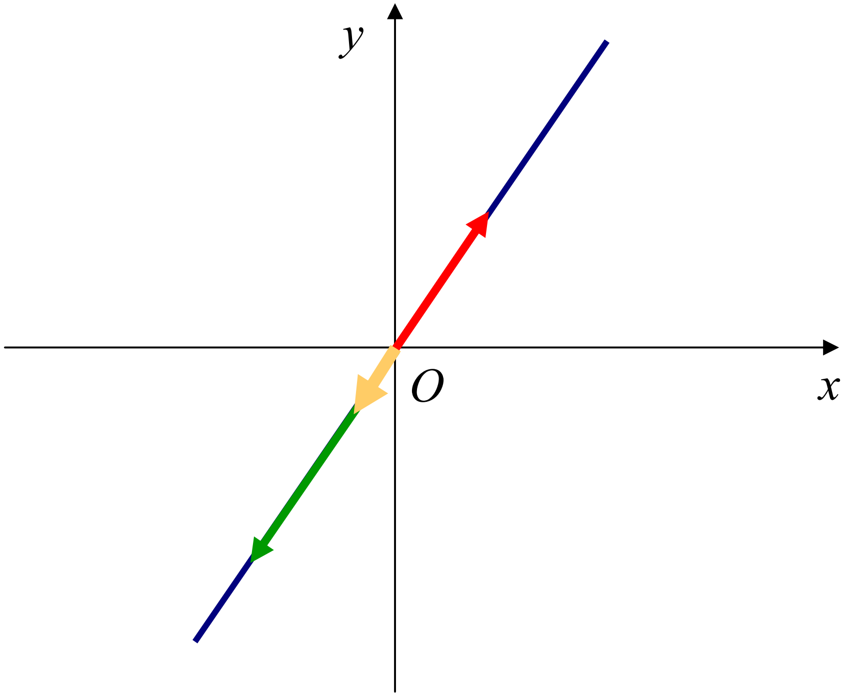
- Informal definition:
  - $V \neq \varnothing$ (a non-empty set of vectors)
  - $\mathbf{v}, \mathbf{w} \in V \;\Rightarrow\; \mathbf{v} + \mathbf{w} \in V$ (closed under addition)
  - $\mathbf{v} \in V$, α is scalar $\;\Rightarrow\; \alpha\mathbf{v} \in V$ (closed under multiplication by scalar)

- Formal definition includes axioms about associativity and distributivity of the + and · operators.

- $0 \in V$ always!

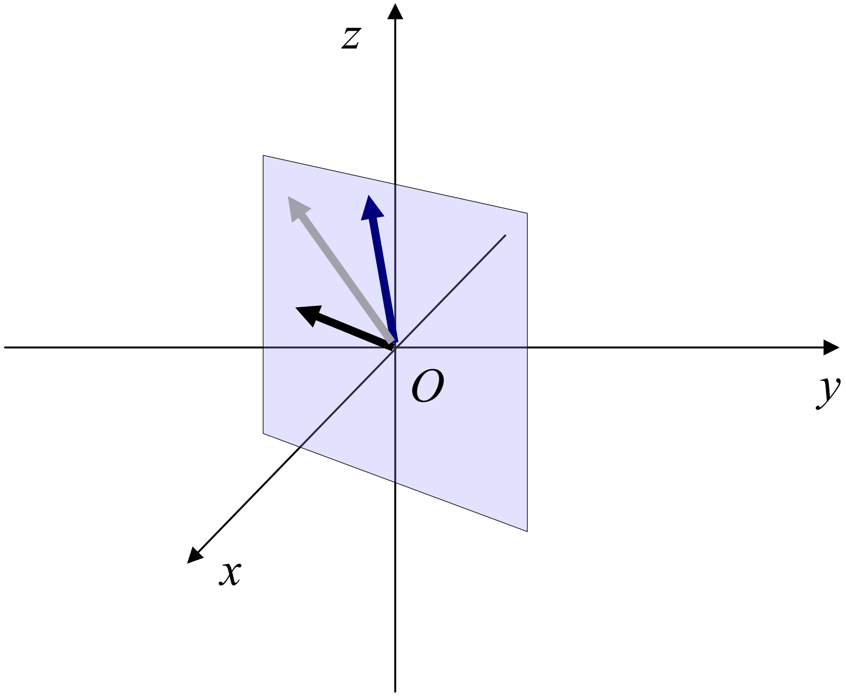
Subspace - example

- Let l be a 2D line though the origin
- $L = \{\mathbf{p} - O \mid \mathbf{p} \in l\}$ is a linear subspace of R^2



Subspace - example

- Let π be a plane through the origin in 3D
- $V = \{\mathbf{p} - O \mid \mathbf{p} \in \pi\}$ is a linear subspace of R^3



Linear independence

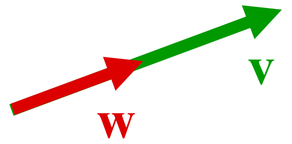
- The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\mathrm{k}\}$ are a linearly independent set if:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_\mathrm{k} = 0 \quad \Leftrightarrow \quad \alpha_i = 0 \;\forall i$$

- It means that none of the vectors can be obtained as a linear combination of the others.

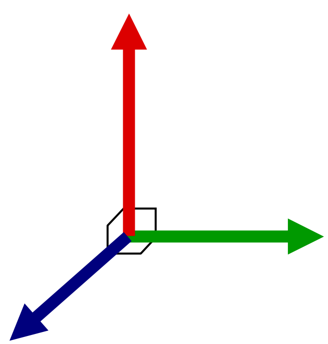
Linear independence - example

- Parallel vectors are always dependent:



$$\mathbf{v} = 2.4\,\mathbf{w} \quad \Rightarrow \quad \mathbf{v} + (-2.4)\mathbf{w} = 0$$

- Orthogonal vectors are always independent.



Basis of V

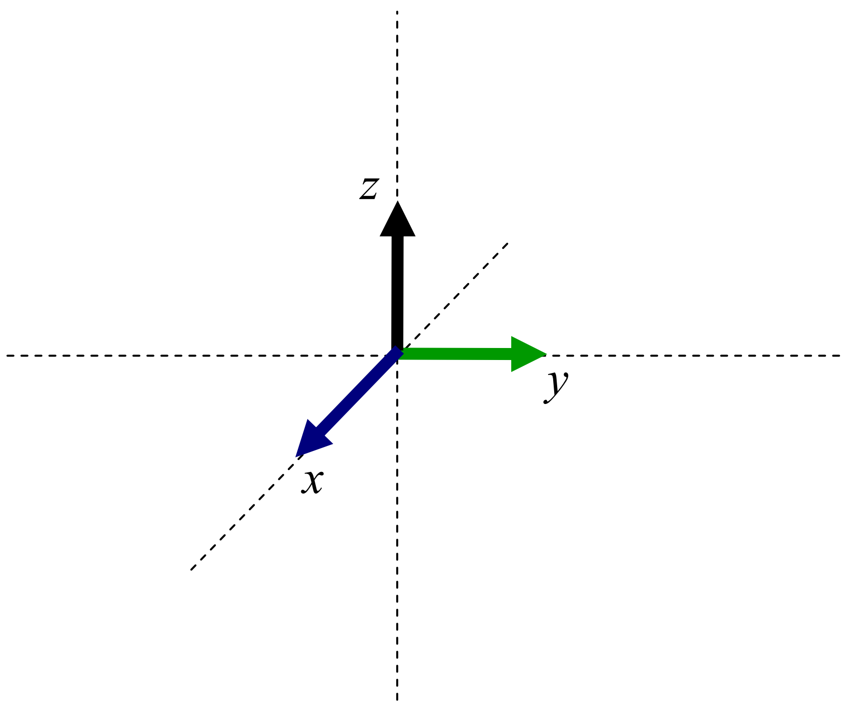
- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly independent
- $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ span the whole vector space V:

$$V = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n \mid \alpha_i \text{ is scalar}\}$$

- Any vector in V is a unique linear combination of the basis.
- The number of basis vectors is called the dimension of V.

Basis - example

- The standard basis of R^3 – three unit orthogonal vectors $\hat{x}, \hat{y}, \hat{z}$: (sometimes called $\hat{i}, \hat{j}, \hat{k}$ or e_1, e_2, e_3)



Matrix representation

- Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of V
- Every $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

- Denote $\mathbf{v}$ by the column-vector:

$$\mathbf{v} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

- The basis vectors are therefore denoted:

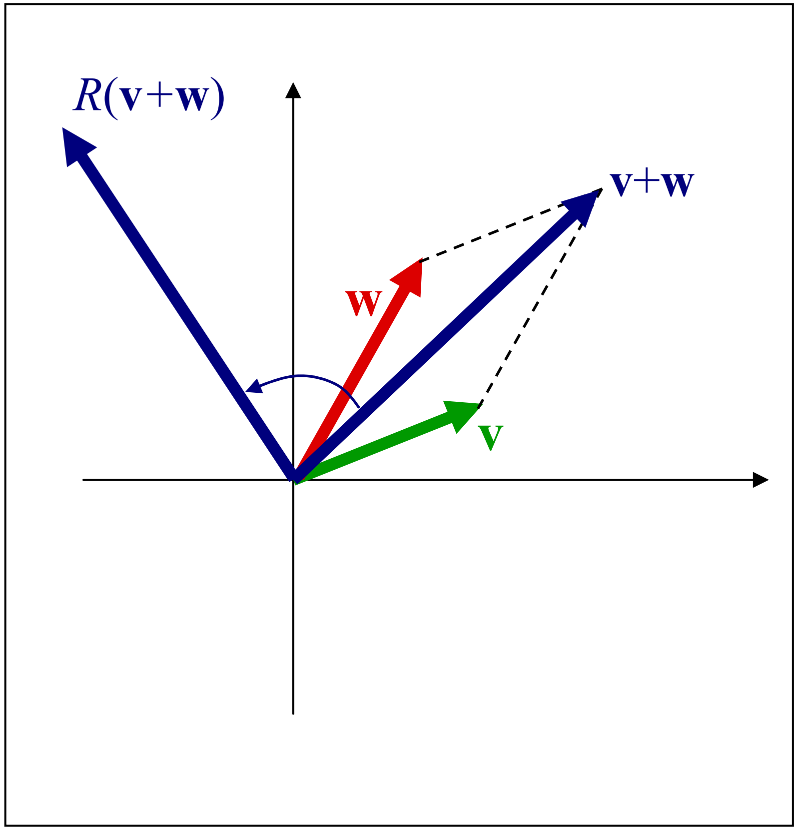
$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

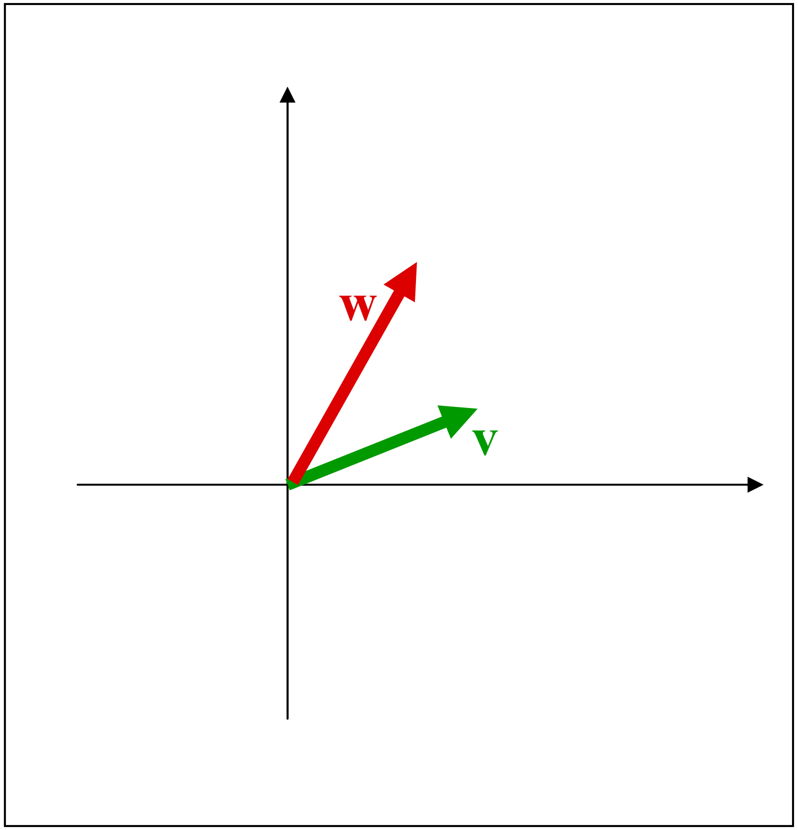
Linear operators

- $A : V \to W$ is called linear operator if:
  - $A(\mathbf{v} + \mathbf{w}) = A(\mathbf{v}) + A(\mathbf{w})$
  - $A(\alpha\,\mathbf{v}) = \alpha\,A(\mathbf{v})$
- In particular, $A(0) = 0$
- Linear operators we know:
  - Scaling
  - Rotation, reflection
  - Translation is not linear – moves the origin

Linear operators - illustration

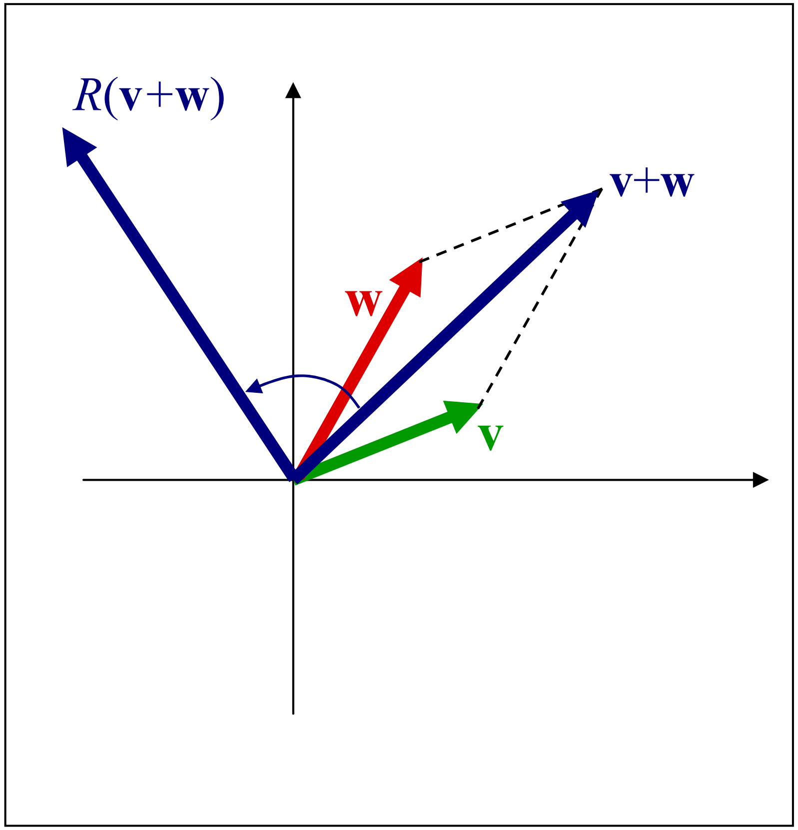
- Rotation is a linear operator:

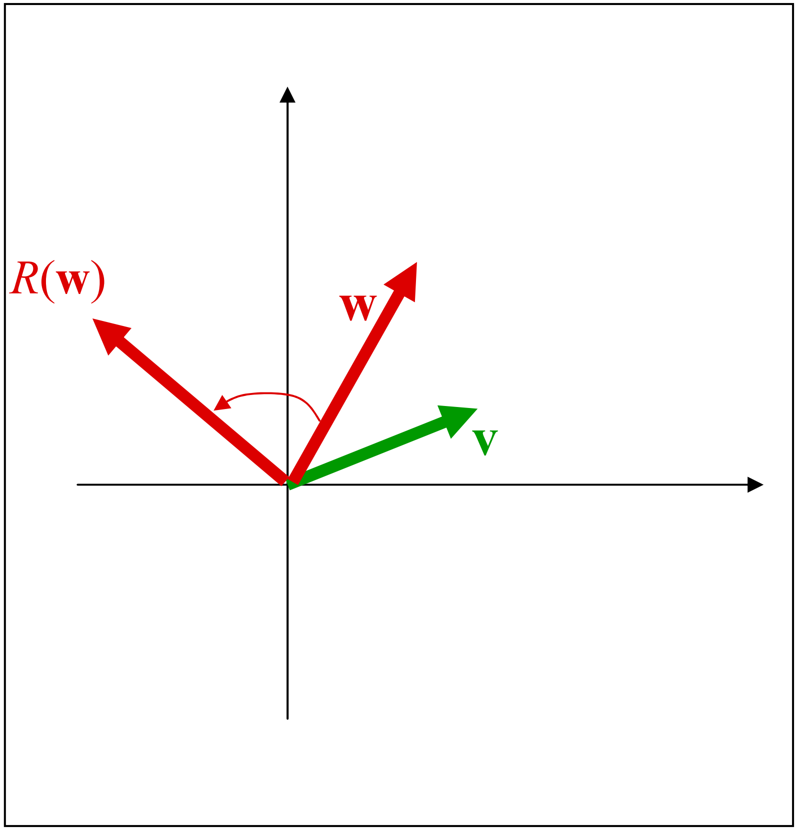




Linear operators - illustration

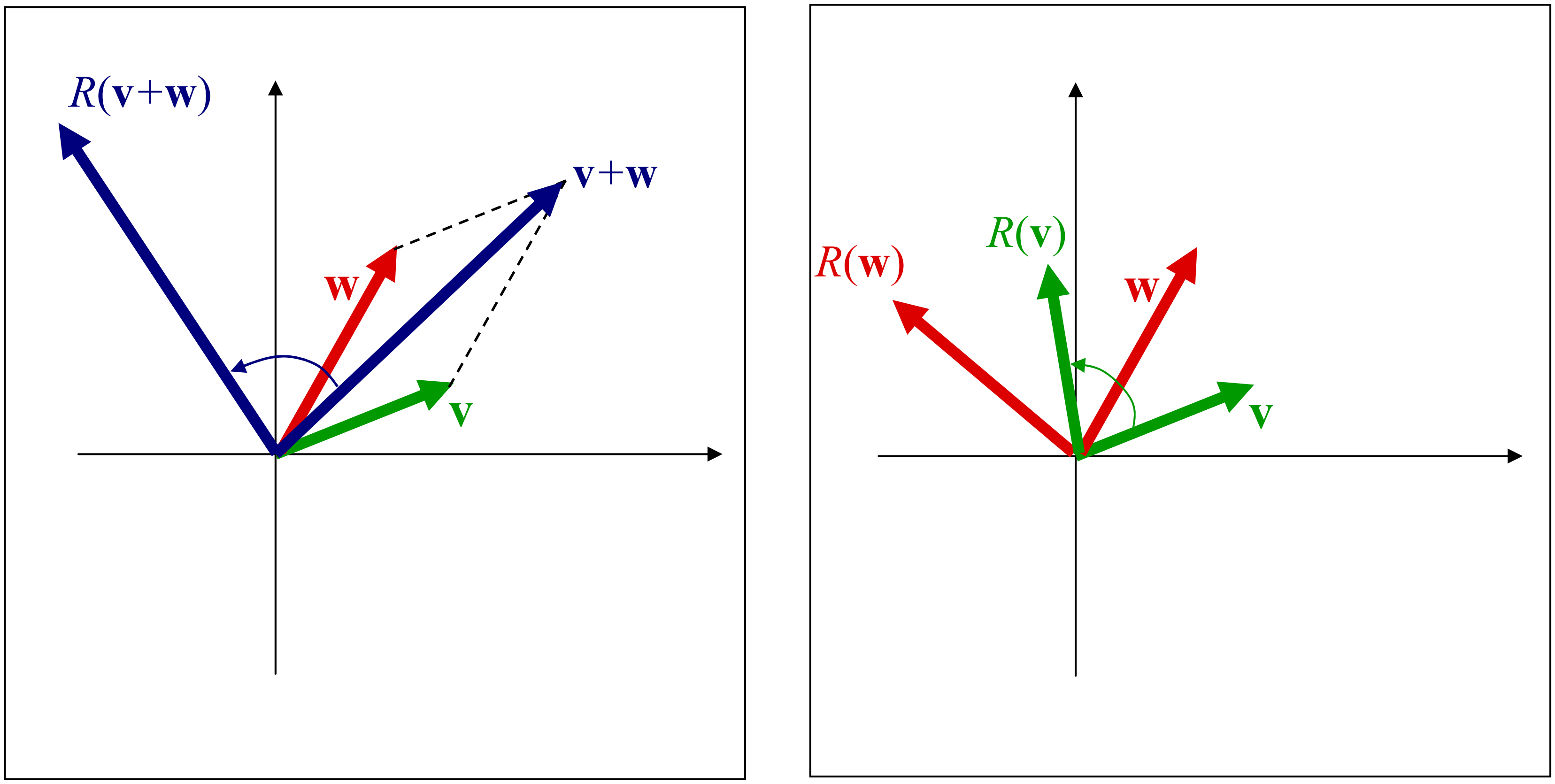
- Rotation is a linear operator:





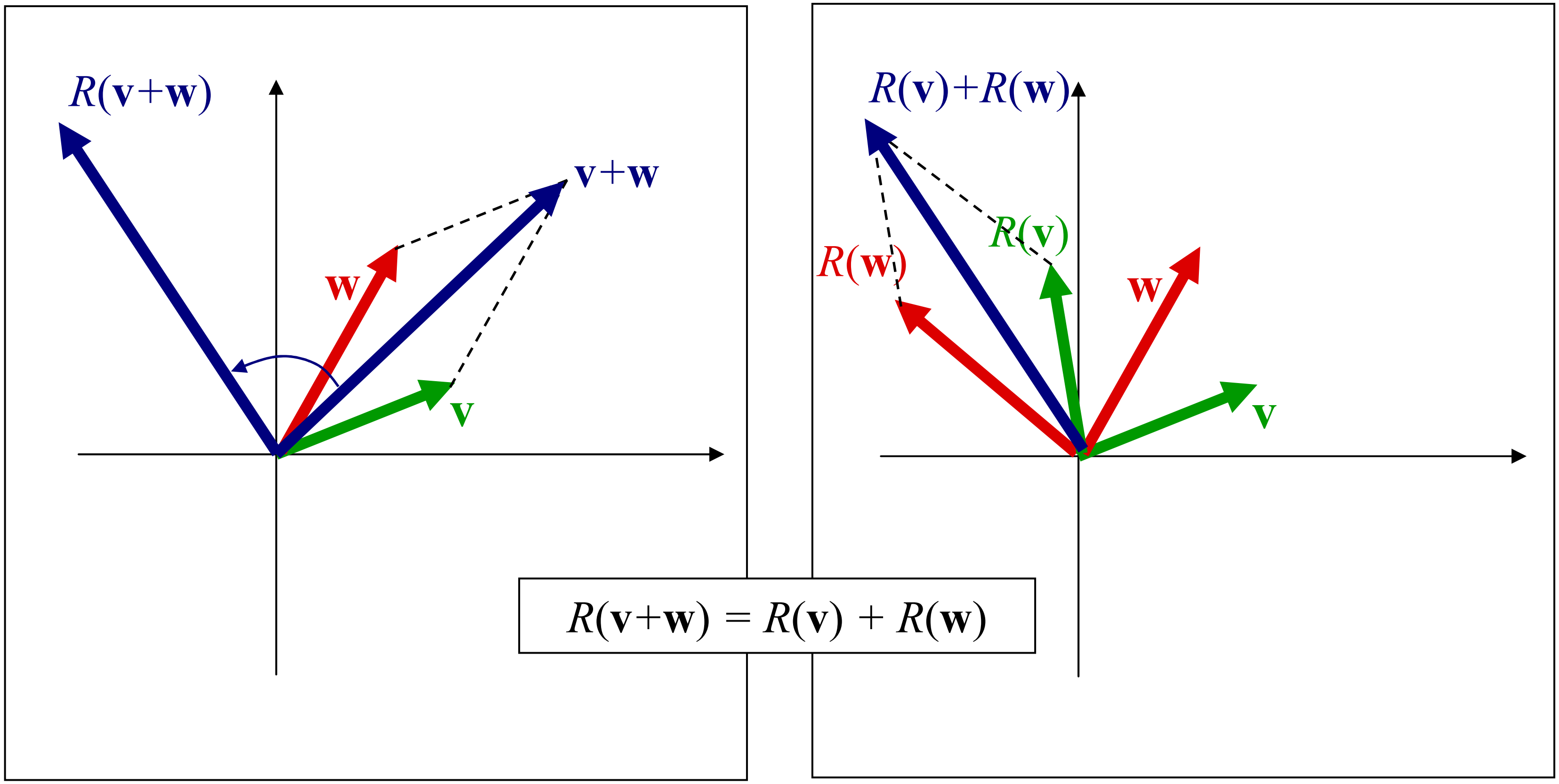
Linear operators - illustration

- Rotation is a linear operator:



Linear operators - illustration

- Rotation is a linear operator:



$$R(\mathbf{v}+\mathbf{w}) = R(\mathbf{v}) + R(\mathbf{w})$$

Matrix representation of linear operators

- Look at $A(\mathbf{v}_1), \ldots, A(\mathbf{v}_n)$ where $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis.
- For all other vectors:

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

$$A(\mathbf{v}) = \alpha_1 A(\mathbf{v}_1) + \alpha_2 A(\mathbf{v}_2) + \ldots + \alpha_n A(\mathbf{v}_n)$$

- So, knowing what A does to the basis is enough
- The matrix representing A is:

$$M_A = \begin{pmatrix} | & | & & | \\ A(\mathbf{v}_1) & A(\mathbf{v}_2) & \cdots & A(\mathbf{v}_n) \\ | & | & & | \end{pmatrix}$$

Matrix representation of linear operators

$$\begin{pmatrix} | & | & & | \\ A(\mathbf{v}_1) & A(\mathbf{v}_2) & \cdots & A(\mathbf{v}_n) \\ | & | & & | \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} | \\ A(\mathbf{v}_1) \\ | \end{pmatrix}$$

$$\begin{pmatrix} | & | & & | \\ A(\mathbf{v}_1) & A(\mathbf{v}_2) & \cdots & A(\mathbf{v}_n) \\ | & | & & | \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} | \\ A(\mathbf{v}_2) \\ | \end{pmatrix}$$

Matrix operations

- Addition, subtraction, scalar multiplication – simple…
- Multiplication of matrix by column vector:

$$\underset{A}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}} \underset{\mathbf{b}}{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}} = \begin{pmatrix} \sum_i a_{1i} b_i \\ \vdots \\ \sum_i a_{mi} b_i \end{pmatrix} = \begin{pmatrix} < \mathrm{row}_1, \mathbf{b} > \\ \vdots \\ < \mathrm{row}_m, \mathbf{b} > \end{pmatrix}$$

Matrix by vector multiplication

- Sometimes a better way to look at it:
  - $A\mathbf{b}$ is a linear combination of A's *columns*!

$$\begin{pmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1 \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} + b_2 \begin{pmatrix} | \\ \mathbf{a}_2 \\ | \end{pmatrix} + \ldots + b_n \begin{pmatrix} | \\ \mathbf{a}_n \\ | \end{pmatrix}$$

Matrix operations

- Transposition: make the rows to be the columns

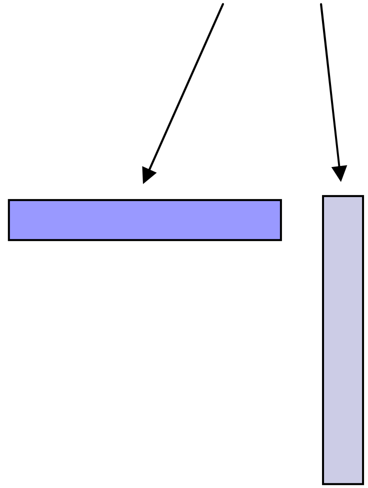
$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$$

- $(AB)^T = B^T A^T$

Matrix operations

- Inner product can in matrix form:

$$< \mathbf{v}, \mathbf{w} > = \mathbf{v}^T \mathbf{w} = \mathbf{w}^T \mathbf{v}$$



Matrix properties

- Matrix A $(n\times n)$ is non-singular if $\exists B,\ AB = BA = I$
- $B = A^{-1}$ is called the inverse of A
- A is non-singular $\Leftrightarrow$ $det\ A \neq 0$

- If A is non-singular then the equation $A\mathbf{x}=\mathbf{b}$ has one unique solution for each $\mathbf{b}$.
- A is non-singular $\Leftrightarrow$ the rows of A are linearly independent (and so are the columns).

Orthogonal matrices

- Matrix A $(n \times n)$ is orthogonal if $A^{-1} = A^T$
- Follows: $AA^T = A^T A = I$
- The rows of A are orthonormal vectors!

Proof:

$$I = A^T A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \mathbf{v}_i^T \mathbf{v}_j \end{pmatrix} = \begin{pmatrix} \delta_{ij} \end{pmatrix}$$

$$\Rightarrow \langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1 \quad \Rightarrow \quad \|\mathbf{v}_i\| = 1; \quad \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$

Orthogonal operators

- A is orthogonal matrix $\Rightarrow$ A represents a linear operator that preserves inner product (i.e., preserves lengths and angles):

$$< A\mathbf{v}, A\mathbf{w} >= (A\mathbf{v})^T (A\mathbf{w}) = \mathbf{v}^T A^T A\mathbf{w} =$$

$$= \mathbf{v}^T I \mathbf{w} = \mathbf{v}^T \mathbf{w} =< \mathbf{v}, \mathbf{w} > .$$

- Therefore, $||A\mathbf{v}|| = ||\mathbf{v}||$ and $\angle(A\mathbf{v}, A\mathbf{w}) = \angle(\mathbf{v}, \mathbf{w})$

Orthogonal operators - example

- Rotation by α around the z-axis in R^3:

$$R = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- In fact, any orthogonal 3×3 matrix represents a rotation around some axis and/or a reflection
  - $\det A = +1$ rotation only
  - $\det A = -1$ with reflection

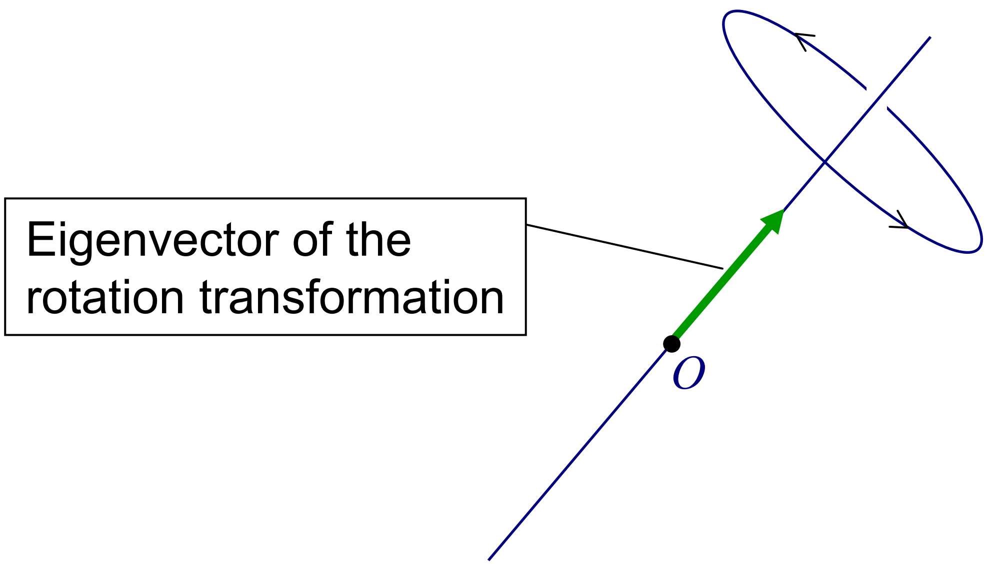
Eigenvectors and eigenvalues

- Let A be a square $n \times n$ matrix
- $\mathbf{v}$ is eigenvector of A if:
  - $A\mathbf{v} = \lambda\mathbf{v}$ (λ is a scalar)
  - $\mathbf{v} \neq 0$
- The scalar λ is called eigenvalue

- $A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A(\alpha\mathbf{v}) = \lambda(\alpha\,\mathbf{v}) \Rightarrow \alpha\,\mathbf{v}$ is also eigenvector
- $A\mathbf{v} = \lambda\mathbf{v},\ A\mathbf{w} = \lambda\mathbf{w} \Rightarrow A(\mathbf{v}+\mathbf{w}) = \lambda(\mathbf{v}+\mathbf{w})$
- Therefore, eigenvectors of the same λ form a linear subspace.

Eigenvectors and eigenvalues

- An eigenvector spans an axis (subspace of dimension 1) that is invariant to A – it remains the same under the transformation.
- Example – the axis of rotation:



Finding eigenvalues

- For which λ is there a non-zero solution to $A\mathbf{x} = \lambda\mathbf{x}$?
- $A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow A\mathbf{x} - \lambda\mathbf{x} = 0 \Leftrightarrow A\mathbf{x} - \lambda I\mathbf{x} = 0 \Leftrightarrow (A-\lambda I)\,\mathbf{x} = 0$
- So, non trivial solution exists $\Leftrightarrow \det(A - \lambda I) = 0$

- $\Delta_A(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n.
- It is called the characteristic polynomial of A.
- The roots of Δ_A are the eigenvalues of A.
- Therefore, there are always at least complex eigenvalues. If n is odd, there is at least one real eigenvalue.

Example of computing Δ_A

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & -3 \\ -1 & 1 & 4 \end{pmatrix}$$

$$\Delta_A(\lambda) = \det\begin{pmatrix} 1-\lambda & 0 & 2 \\ 3 & -\lambda & -3 \\ -1 & 1 & 4-\lambda \end{pmatrix} =$$

$$= (1-\lambda)(-\lambda(4-\lambda)+3) + 2(3-\lambda) = (1-\lambda)^2(3-\lambda) + 2(3-\lambda) =$$

$$= (3-\lambda)(\lambda^2 - 2\lambda + 3)$$

Cannot be factorized over R
Over C: $(1+i\sqrt{2})(1-i\sqrt{2})$

Computing eigenvectors

- Solve the equation $(A - \lambda I)\mathbf{x} = 0$
- We'll get a subspace of solutions.

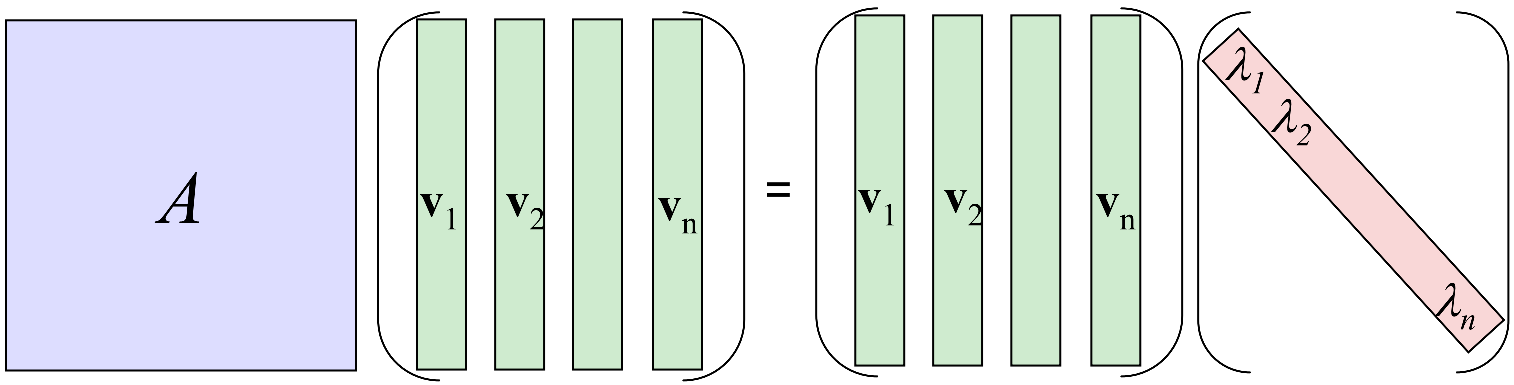
Spectra and diagonalization

- The set of all the eigenvalues of A is called the spectrum of A.

$$AV = VD$$

- A is diagonalizable if A has n independent eigenvectors. Then:

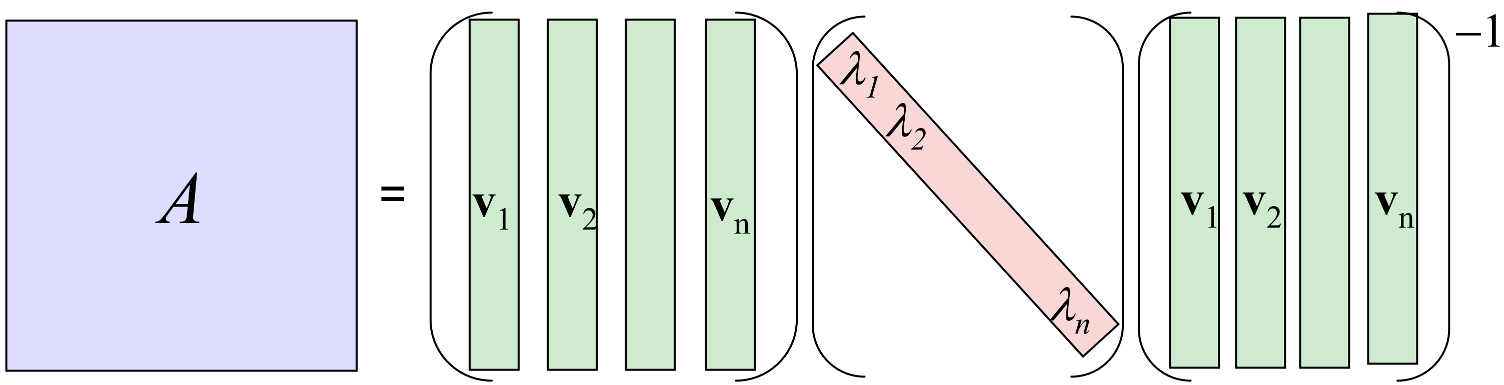
$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$$
$$A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$$
$$\vdots$$
$$A\mathbf{v}_n = \lambda_n \mathbf{v}_n$$



Spectra and diagonalization

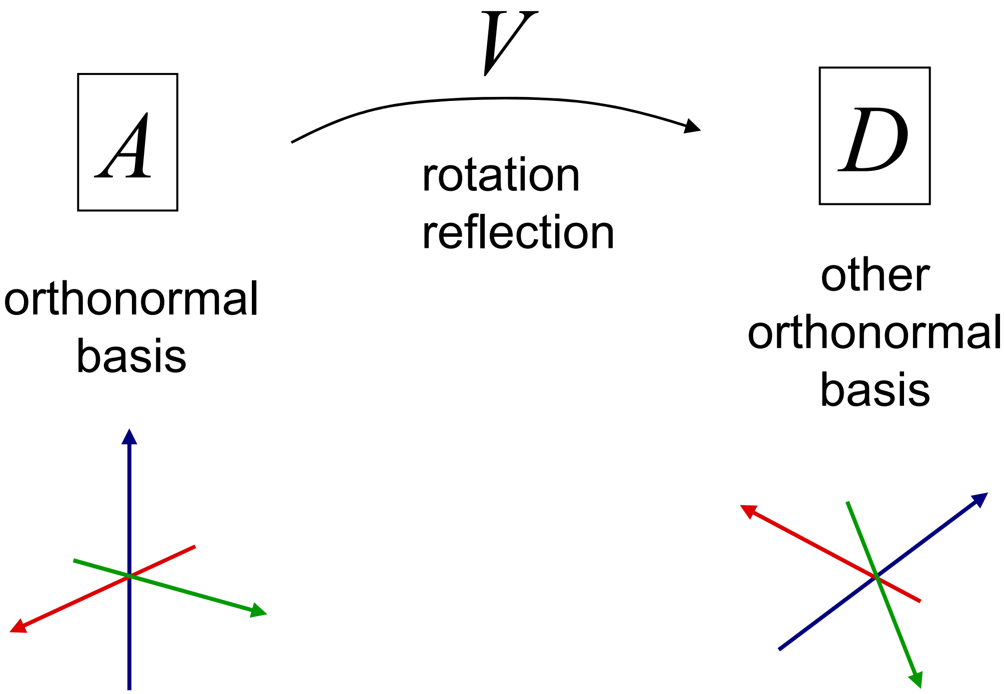
- Therefore, $A = \mathrm{V}D\mathrm{V}^{-1}$, where D is diagonal
- A represents a scaling along the eigenvector axes!

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$$
$$A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$$
$$\vdots$$
$$A\mathbf{v}_n = \lambda_n\mathbf{v}_n$$

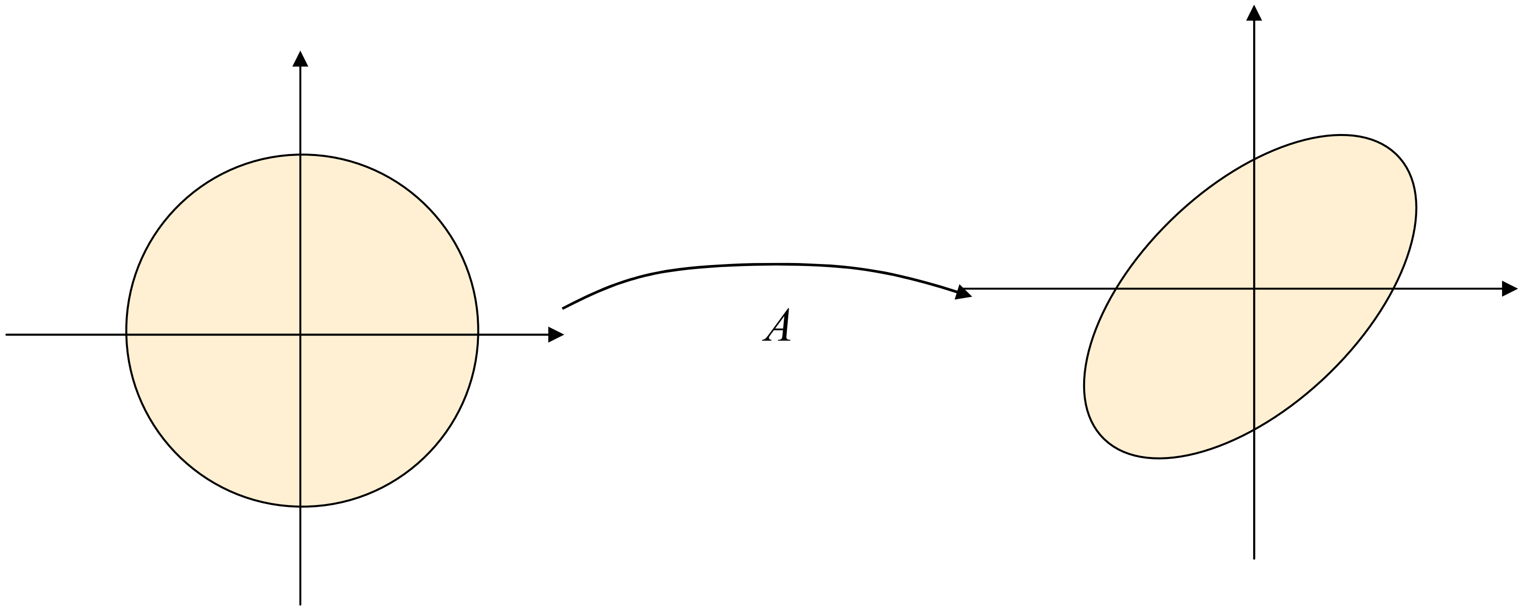


$$A = \mathrm{V}D\mathrm{V}^{-1}$$

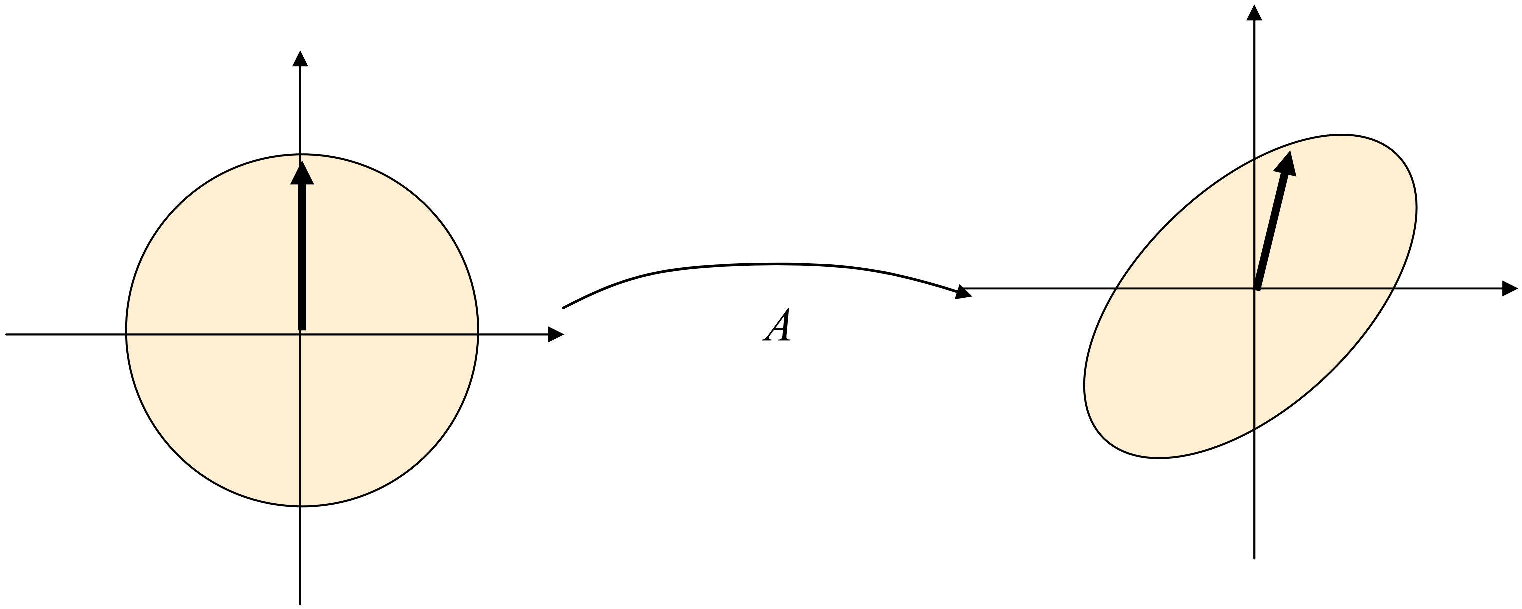
Spectra and diagonalization



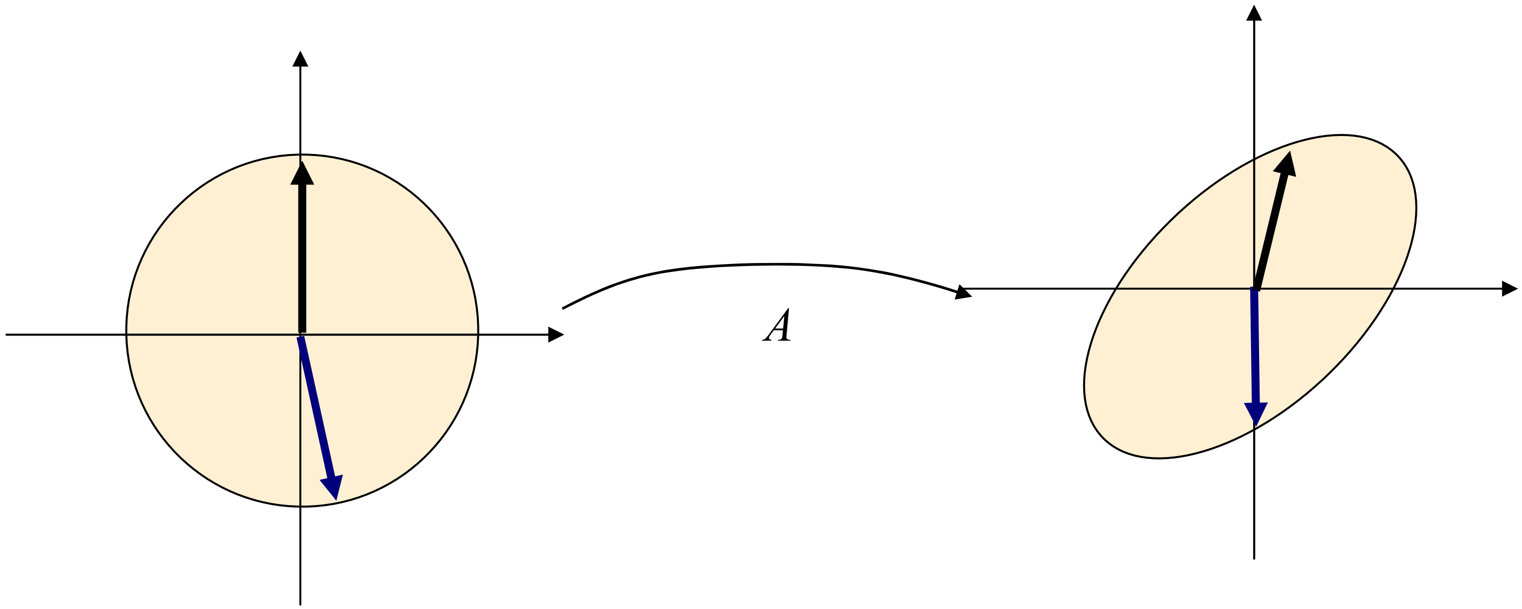
Spectra and diagonalization



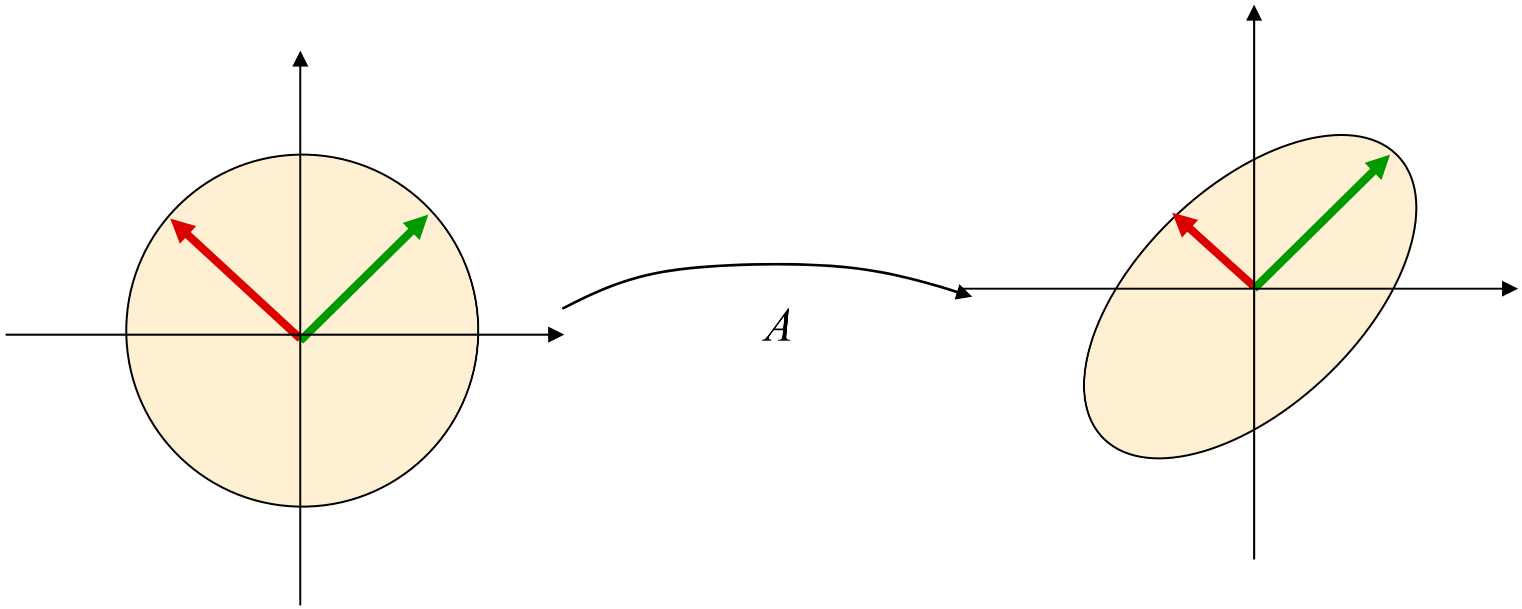
Spectra and diagonalization



Spectra and diagonalization



Spectra and diagonalization



eigenbasis

Spectra and diagonalization

- A is called normal if $AA^T = A^TA$.
- Important example: symmetric matrices $A = A^T$.
- It can be proved that normal $n \times n$ matrices have exactly n linearly independent eigenvectors (over C).
- If A is symmetric, all eigenvalues of A are real, and it has n independent real orthonormal eigenvectors.

Why SVD…

- Diagonalizable matrix is essentially a scaling.
- Most matrices are not diagonalizable – they do other things along with scaling (such as rotation)

- So, to understand how general matrices behave, only eigenvalues are not enough
- SVD tells us how general linear transformations behave, and other things…

Next Week: Mysteries of the PCA & SVD