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# Causal Graph Dynamics

[ICALP 2012, I&C 2013, arXiv:1202.1098]

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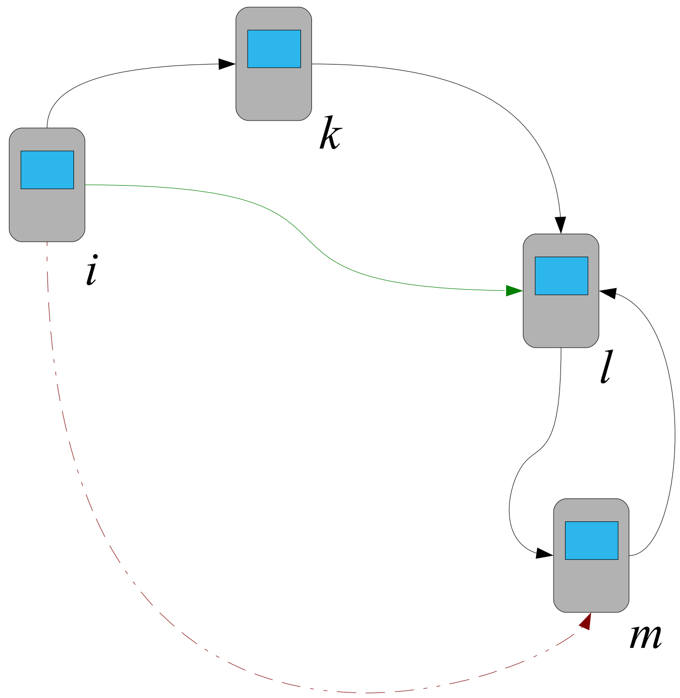
# Problem > Understanding the causality property

Ex. Mobile phone network

| | | |
|---|---|---|
| Phones | $\hookrightarrow$ | vertices, |
| Contacts | $\hookrightarrow$ | Edges, |
| Internal states | $\hookrightarrow$ | Labels, |
| Call duration | $\hookrightarrow$ | One time step... |




Moreover:

- Phones maybe created
- Or thrown out

… naturally the graph evolves *causally*, but...
...try define it!

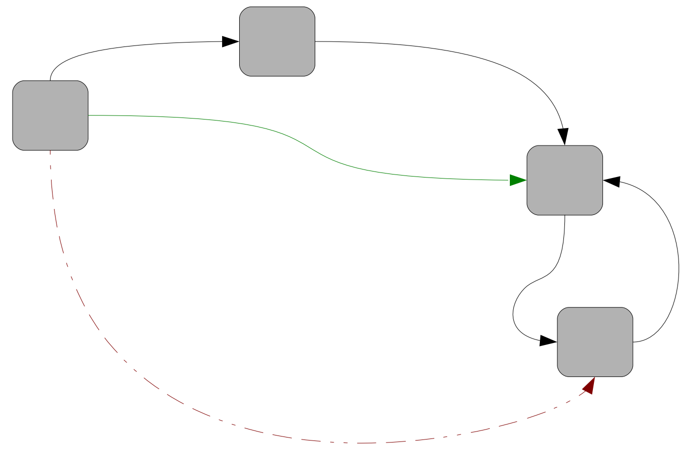
Other

Social networks, epidemiology, Regge calculus...

## Problem > Understanding the causality property

Problem: The double role of the notion of Neighbourhood

- Neighbourhood is a constraint upon the evolution
- Neighbourhood is a subject of the evolution



Problem: The notion of antecedent
Needed to state causality.
*Ex: New state of a new mobile depends only on neighbours of... whom?*

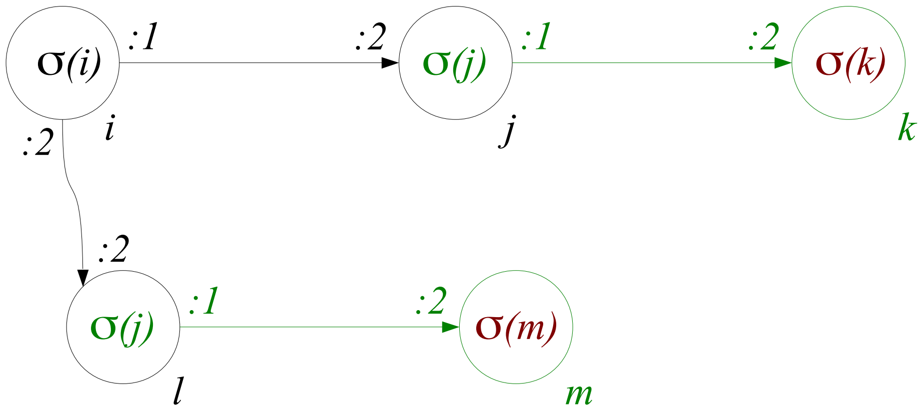
Problem: The notion of translation invariance
Make vertex names made irrelevant... yet they are useful!
*Ex: Your behaviour is independent of your phone number.*

Graphs? In $\mathcal{G}_{\Sigma,\pi}$ :


Neighbourhoods?

$G_i^0$

$G_i^1$ ...



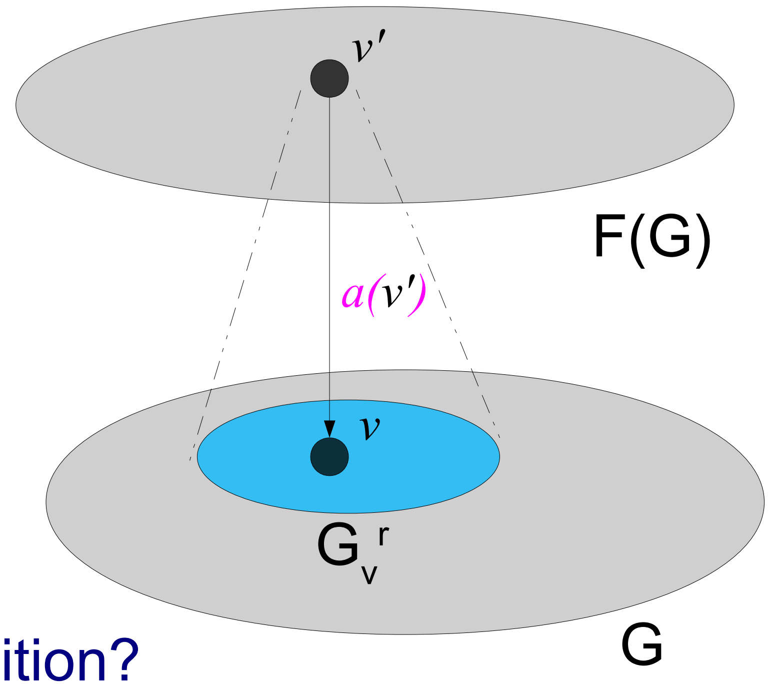

So, when is a dynamics F: $\mathcal{G}_{\Sigma,\Delta,\pi} \rightarrow \mathcal{G}_{\Sigma,\Delta,\pi}$ causal?

A dynamics F: $\mathcal{G}_{\Sigma,\pi} \rightarrow \mathcal{G}_{\Sigma,\pi}$ is causal iff

$\exists r, \forall v', v \in a(v'), \forall G, H,$

$$G_v^r = H_v^r \Rightarrow F(G)_{v'}^0 = F(H)_{v'}^0$$

*i.e. the state and connectivity of v' depends on the neighbourhood of one of its antecedents.*




Physical, axiomatic, no-signalling condition.
But abstract mathematical object.

Do we have a more concrete alternative definition?

## Mathematical definition > **Local rule-induced graph dynamics**

A dynamics F: $\mathcal{G}_{\Sigma,\pi} \rightarrow \mathcal{G}_{\Sigma,\pi}$ is causal iff

$\exists r, \forall v', v \in a(v'), \forall G, H, \quad G_v^r = H_v^r \Rightarrow F(G)_{v'}^0 = F(H)_{v'}^0$

The physical, axiomatic, no-signalling condition.

But abstract mathematical object.

A dynamics F: $\mathcal{G}_{\Sigma,\pi} \rightarrow \mathcal{G}_{\Sigma,\pi}$ is localizable iff

$\exists r, \exists f, \forall G,$

$$F(G) = \bigcup_v f(G_v^r)$$

Union of G and H? OK if they G and H are consistent i.e.:

- $\sigma_G$ and $\sigma_H$ do not disagree upon $G \cap H$
- $E_G$ and $E_H$ do not disagree upon $G \cap H$

## Mathematical definition > **Local rule-induced graph dynamics**

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But abstract mathematical object.

A dynamics F: $\mathcal{G}_{\Sigma,\pi} \rightarrow \mathcal{G}_{\Sigma,\pi}$ is localizable iff

$\exists r, \exists f$ a local rule, $\forall G$,

$$F(G) = \bigcup_v f(G_v^r)$$

with f local rule, i.e a dynamics with consistent images.

Concrete, constructive, plausible definition.

But an ad-hoc construction?

Are the two equivalent?

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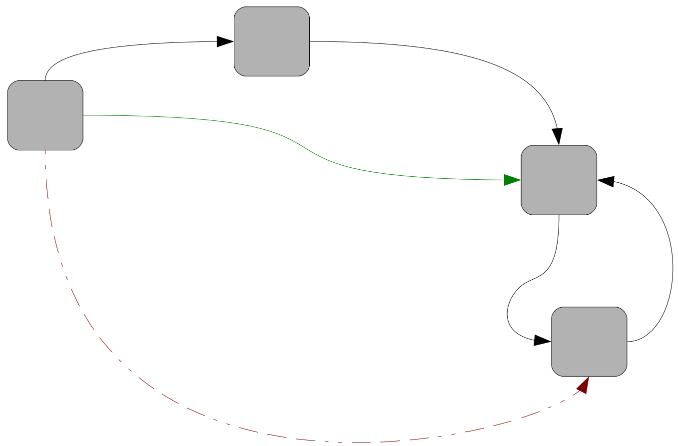


Problem: The notion of antecedent

Needed to state causality.

*Ex: New state of a new mobile depends only on neighbours of... whom?*

Problem: The notion of dynamics, i.e. "translation invariance"

Make vertex names made irrelevant... yet they are useful!

*Ex: Your behaviour is independent of your phone number.*

# Problem > Understanding the “translation-invariance” property

Difficulty: Make vertex names made irrelevant... but they are useful!

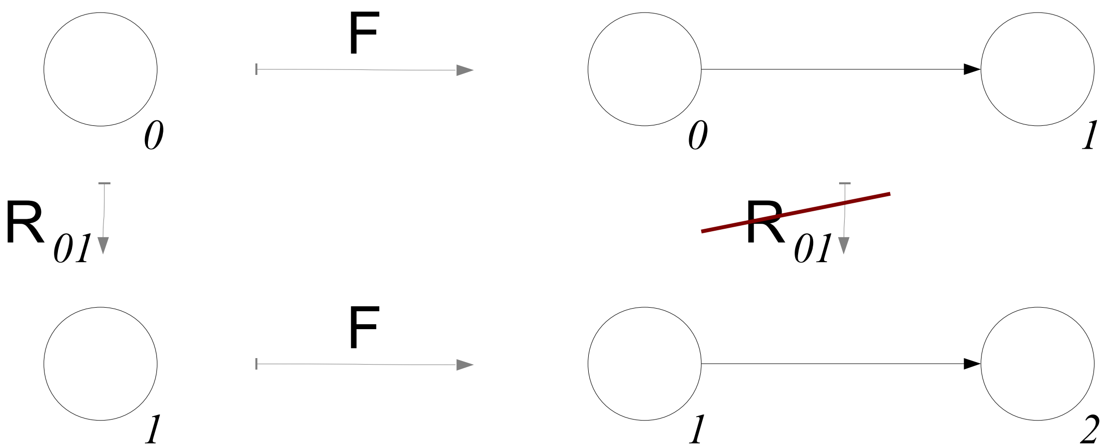
Approach 1: Graphs modulo isomorphisms, but:

- no way to designate a vertex
- no notion of union
- no notion of antecedent

Approach 2: Evolutions commute with isomorphisms

$\forall R,\ \ R \circ F = F \circ R$

But: no possible node creation:



## Problem > Understanding the “translation-invariance” property

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Approach 2: Evolutions commute with isomorphisms

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But: no possible node creation.

Notice this commmutation implies:

- Conjugacy: $\forall R, \exists R' /\ R' \circ F = F \circ R$
- Freshness: $\cap G^{(i)} = \varnothing \Rightarrow \cap F(G^{(i)}) = \varnothing$

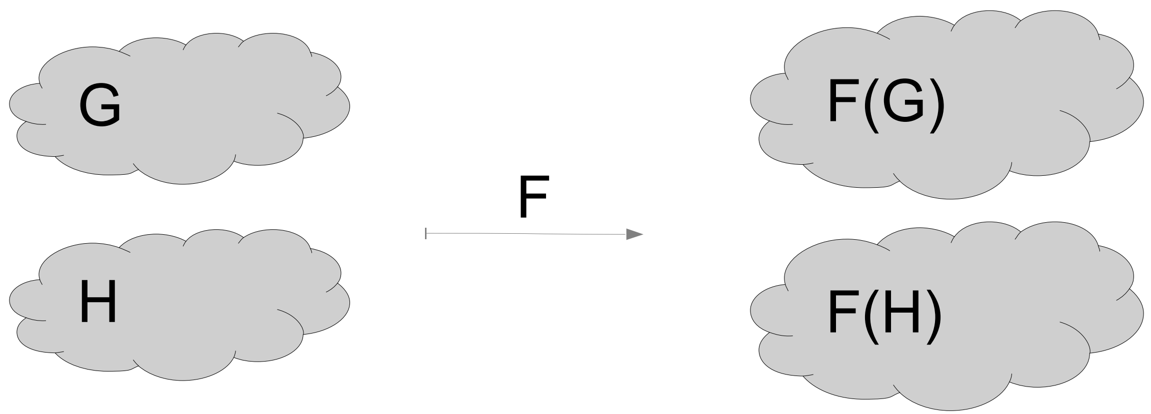
Graphs Dynamics

(i) Conjugacy: $\forall R, \exists R' \,/\, R' \circ F = F \circ R$

(ii) Freshness: $\bigcap G^{(i)} = \emptyset \Rightarrow \bigcap F(G^{(i)}) = \emptyset$

Why Freshness?

- Also a weakening of commutation
- Disconnected universes remain so
- $F(\emptyset) = \emptyset$




Is a solution:

Makes node names made somewhat irrelevant... and yet useful!

Graphs Dynamics...

(i) Conjugacy: $\forall R, \exists R' / R'F = FR$

(ii)Freshness: $G \cap H = \emptyset \Rightarrow F(G) \cap F(H) = \emptyset$

...admit a notion of antecedent...

$v \in a(v') \Leftrightarrow [\ \forall G, v' \in F(G) \Rightarrow v \in G\ ]$

*i.e. the ones that give v' its name.*

…which is robust:.

- Co-dynamicity: $R \circ a = a \circ R'$ whenever $R' \circ F = F \circ R$
- $|a(v')| \geq 1$

## Problem > Understanding the causality property

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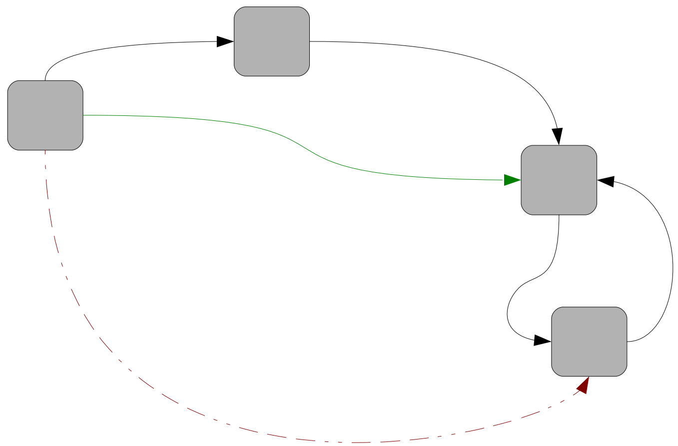

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Physical, axiomatic, no-signalling condition.

A dynamics F: $\mathcal{G}_{\Sigma,\pi} \rightarrow \mathcal{G}_{\Sigma,\pi}$ is localizable iff

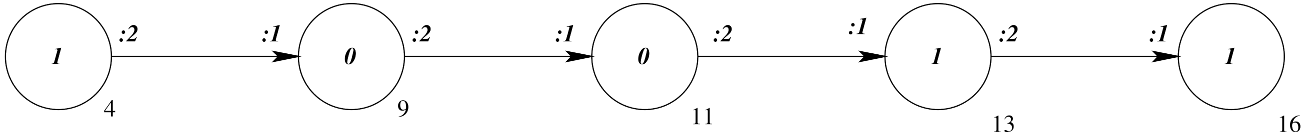
$\exists r, \exists f$ a local rule, $\forall G, \quad F(G) = \cup_v f(G_v^{\,r})$

with f local rule, i.e  a dynamics with consistent images.
Concrete, constructive, plausible definition.

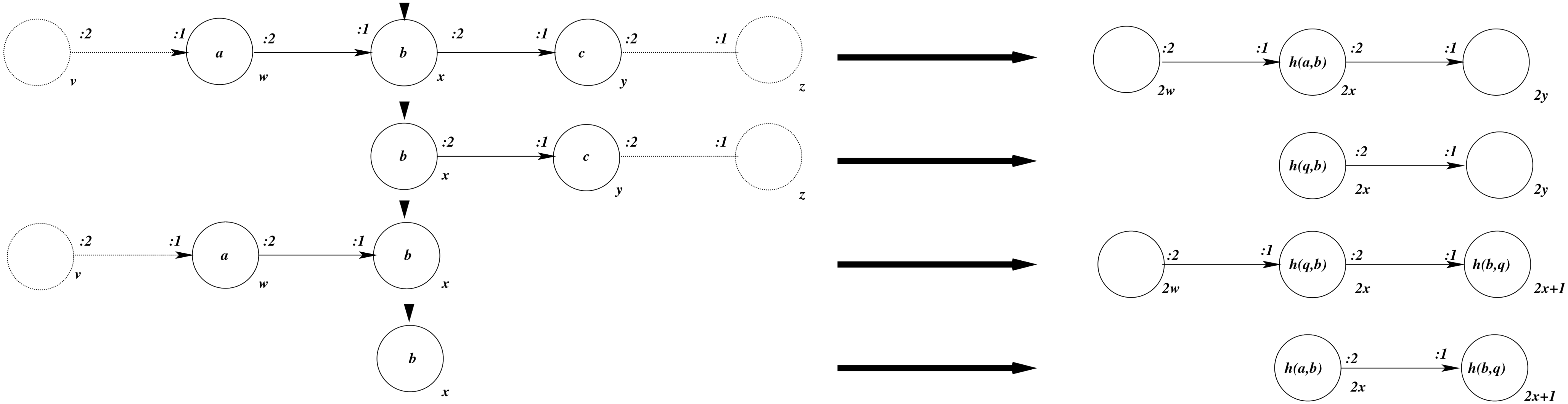
Theorem
A dynamics F: $\mathcal{G}_{\Sigma,\pi} \rightarrow \mathcal{G}_{\Sigma,\pi}$ is causal iff it is localizable.

## Mathematical definition > **Examples > CA**

With configurations ...$qq\Sigma^*qq$... coded by:

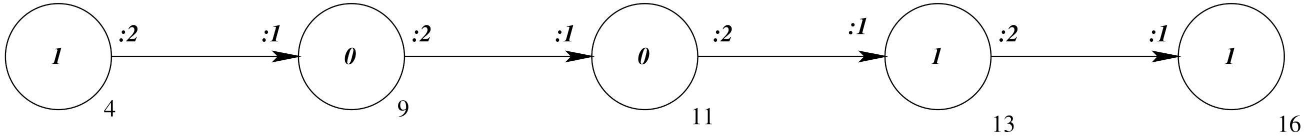



With local rule $h(q,q) = q$ coded by:

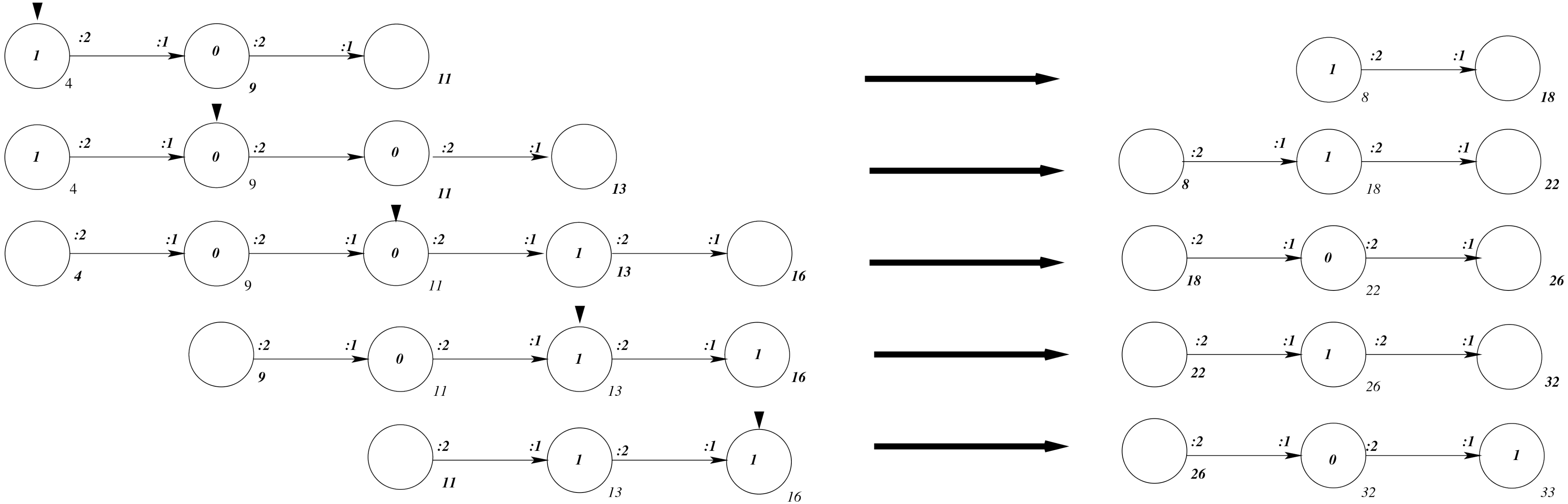



Next, for $q = 0$, $\Sigma = \{0,1\}$ and $h(a,b) = a+b \bmod 2$.

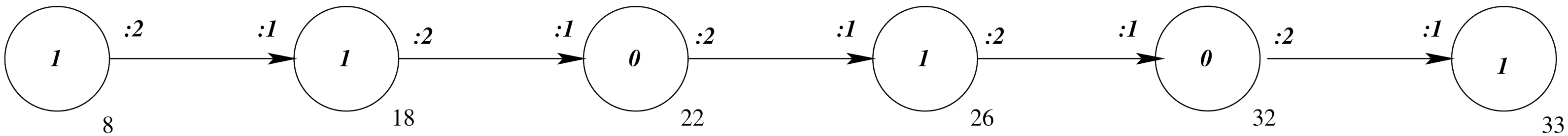
With configuration *...qq10011qq...* and rule *h(a,b)=a+b mod 2*.



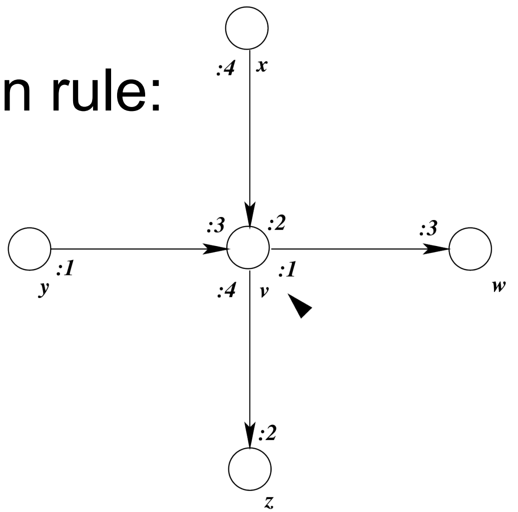

Applying the coded local rule, and glueing...

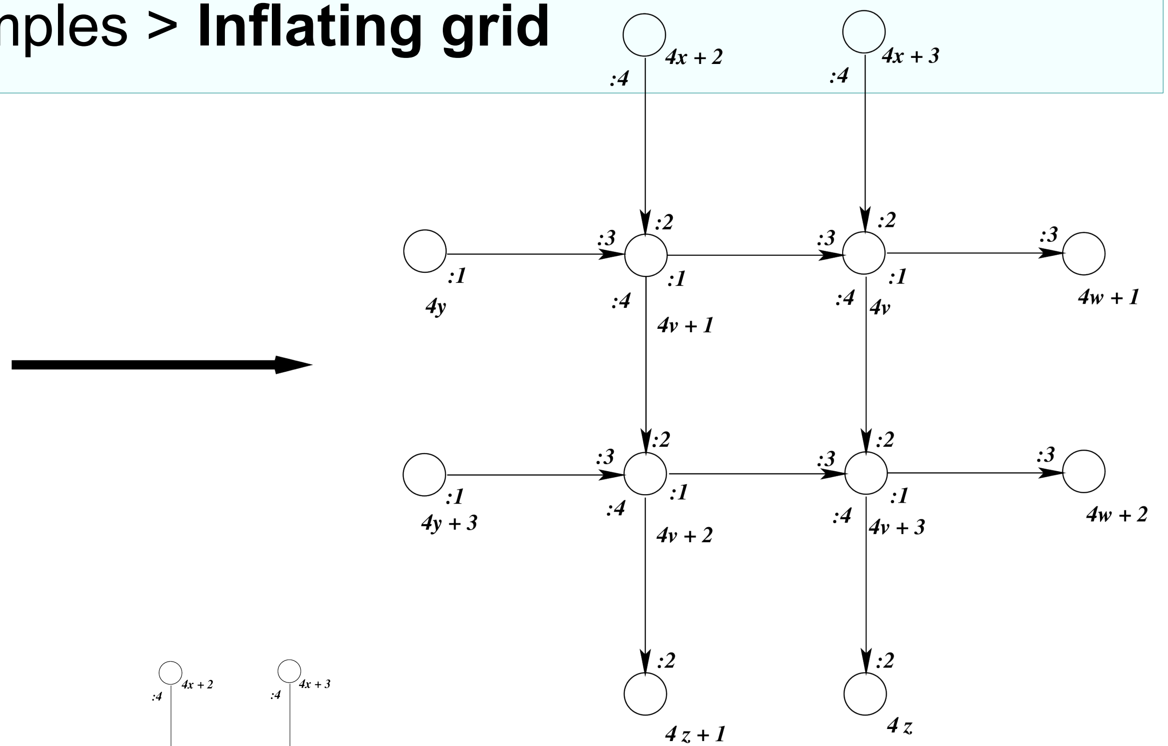


...yields:

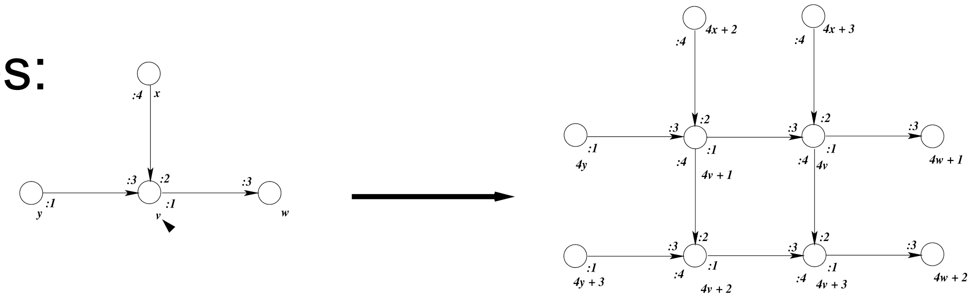


With main rule:

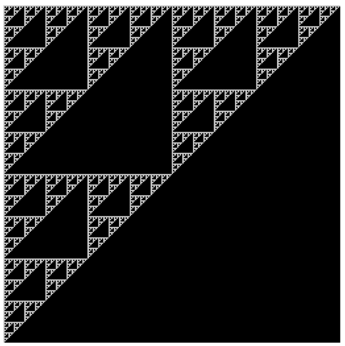


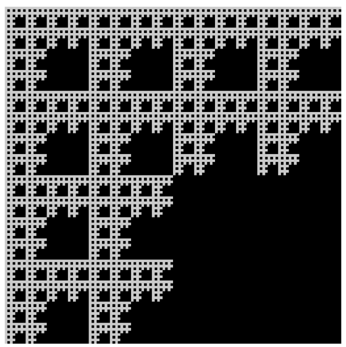




Border cases:




Yields:





## Properties > Stability

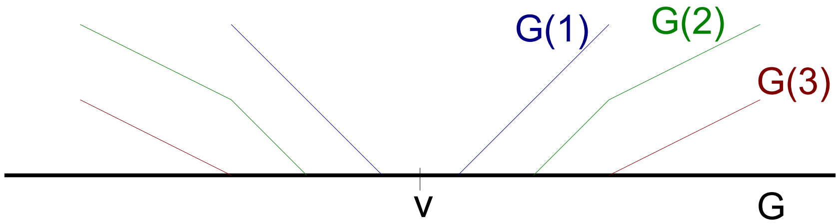
Theorem: Composability
$F_1$ causal and $F_2$ causal implies $F_2 \circ F_1$ causal.

Proposition: Universality of radius one
F causal of radius r can be simulated by some F' causal of radius 1.

## A notion of Limit

The pointed graph sequence $(r \rightarrow (G(r),v))$ converges to $(G,v)$ iff

$$\forall v, \forall s, \exists r \qquad G(r)_v^s = G_v^s$$




## A dynamics F is Limit-preserving iff

$(r \mapsto (G(r),v))$ converges to $(G,v)$ implies
$(r \mapsto (F(G(r)),v))$ converges to $(F(G),a^{-1}(v))$

## A dynamics F is continuous iff

$$\forall r', \forall v', v \in a(v'), \forall G, \exists r, H, \qquad G_v^r = H_v^r \Rightarrow F(G)_{v'}^0 = F(H)_{v'}^0$$

## Proposition: Causal $\Rightarrow$ Continuous $\Leftrightarrow$ Limit-preserving

## Theorem

If $\Sigma$, $\pi$, are finite, Causal $\Leftrightarrow$ Continuous $\Leftrightarrow$ Limit-preserving

A dynamics $F$ is invertible iff

$$\exists F^{-1} \text{ a dynamics } / \ F^{-1}F = FF^{-1} = Id \, .$$

Proposition: Invertible dynamics are connected-preserving

A causal dynamics $F$ is reversible iff
$F$ is invertible with causal $F^{-1}$.

Theorem
If $\Sigma$, $\pi$, are finite,
$F$ causal invertible $\Leftrightarrow$ $F$ causal reversible

## Litterature >

Local dynamics, fixed graph

- CA
- Cayley CA [Roka]
- Graph Automata [Papazian,Remila]

3 definitions:
•Physical (causality)
•Constructive (local rule)
•Mathematical (continuous)

(Local) graph rewriting, fixed labels

- Amalgamated Graph Transformations [Löwe]
- Parallel Graph Transformations [Taentzer]

Many specific purpose models

- Epidemiology [Murray,...]
- Self-reproduction [Tomita,...]

## Conclusion

Done

A notion of causal graph dynamics in three flavours:

- Physical (causal dynamics)
- Constructive (localizable dynamics)
- Mathematical (continuous dynamics)

- Stability under composition
- Stability under inverse
- Universality of radius one

Done also

- A more topological formulation (gen. Cayley graphs) [A., Martiel]
- Causal Dynamics of Discrete surfaces (*2D*) [A., Martiel] *
- Universal Constructions [Martiel, Martin] *

## Future

### Doing

- Causal dynamics of discrete manifolds (*nD*, current)
- More on structure of the reversible case (current)
- The quantum case (static case: tomorrow 12:30 5th floor CPT)
- The probabilistic case.

### Needs be done

- Re-evaluate more CA results in this framework.
- The possibility of simulating isotropic phenomena (FEM...)?