Solution of Exercise 6, Question 4(b)

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Question: You are given an array of size n with $\log n$ distinct integers, each one appearing exactly $\frac{n}{\log n}$ times. Prove a lower bound of $\Omega(n \log\log n)$ in the comparison model for any algorithm which sorts such an array.
Answer: First, assume that the $\log n$ distinct values in the input array are $1, \ldots, \log n$. This does not matter for an algorithm in the comparison model, because such an algorithm only performs comparisons between elements. Now, how many possible inputs are there? If all elements were distinct then the number of inputs would have been $n!$. However, in our case we need to divide by the number of permutations that will keep the array the same. Thus, the number of possible inputs is exactly

$$X = \frac{n!}{(\frac{n}{\log n}!)^{\log n}}$$

Now, let us look at some algorithm in the comparison model, that is a decision tree that performs only comparisons. How many leaves must it have? It is easily seen that a solution can be a correct solution only for one input array. Therefore the algorithm must have at least X leaves. It is a tree where every vertex has 2 or 3 children, and which has at least X leaves, so its depth is at least $\Omega(\log(X))$. Therefore, the running time of any algorithm in the comparison model is at least $\Omega(\log(X))$. Now we wish to prove that $\log(X) = \Omega(n \log\log n)$.

First, let us assume that for every m, $m! = 2^{m \log m}$ and in this case we will easily get that $\log(X) = \Omega(n \log\log n)$. The assumption that $m! = 2^{m\log m}$ is untrue, of course, but the math here is conceptually the right math, and it is cleaner than the correct proof we will give later. So:

$$\log(X) = \log\left(\frac{n!}{(\frac{n}{\log n}!)^{\log n}}\right) = \log(n!) - \log((\frac{n}{\log n}!)^{\log n}) = n\log n - (\log n)\log(\frac{n}{\log n}!) =$$
$$n\log n - (\log n)\frac{n}{\log n}\log(\frac{n}{\log n}) = n\log n - n(\log n - \log\log n) = n\log\log n\ ,$$

as required.

So, how do we formally prove that $\log(X) = \Omega(n\log n)$ in a correct way? We use Stirling's approximation. Stirling's approximation states that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\ .$$

In other words this says that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1\ .$$

But we will be satisfied using the weaker, but still useful expression

$$n! = \Theta\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right) \ .$$

From taking the logarithm of both sides of this expression it follows that

$$\log(n!) = n \log n \pm O(n) \ .$$

We now give a proof that $\log(X) = \Omega(n \log\log n)$:

$$\begin{aligned}
\log(X) = &\log\left(\frac{n!}{(\frac{n}{\log n}!)^{\log n}}\right) = \\
&\log(n!) - \log((\frac{n}{\log n}!)^{\log n}) = n\log n \pm O(n) - (\log n)\log(\frac{n}{\log n}!) = \\
&n\log n \pm O(n) - (\log n)(\frac{n}{\log n}\log(\frac{n}{\log n}) \pm O(\frac{n}{\log n})) = \\
&n\log n \pm O(n) - n(\log n - \log\log n) \pm O(n) = n\log\log n \pm O(n) = \Theta(n\log\log n) \ .
\end{aligned}$$

1 Further Reading

Here are some web pages about lower bounds, which you might find interesting.

1.1 The "P=NP?" Question

http://en.wikipedia.org/wiki/Complexity_classes_P_and_NP
http://www.claymath.org/millennium/P_vs_NP/Official_Problem_Description.pdf
http://www.win.tue.nl/~gwoegi/P-versus-NP.htm
http://www.math.ias.edu/~avi/PUBLICATIONS/MYPAPERS/W06/w06.pdf

1.2 Stirling's Approximation

http://hyperphysics.phy-astr.gsu.edu/Hbase/Math/stirling.html
http://en.wikipedia.org/wiki/Stirling's_approximation

1.3 About lower bounds in computer science

Lecture notes L,H and J in http://compgeom.cs.uiuc.edu/~jeffe/teaching/373/